置换中圈下降位置数的研究

摘要

1749年，欧拉在其报告中首次提出了经典的欧拉多项式。自那以后，欧拉多项式在很多数学领域被广泛地研究。特别地，欧拉多项式有一个有趣的组合学表示方法，即通过置换上的上升位置数来表达。因此，围绕着置换上的上升数，人们做了一些研究，也得到了一些有意义的结果。比如Foata和Brenti的结果就很具有组合意义。然而，当我们初步研究这些问题的时候，我们仍然发现了许多尚未被证明的命题。另外，已经存在的结果可以被细分得更加具体，使得它所具备的组合学意义更加明显。因此，在这篇论文中，我着重研究了置换上的圈下降位置数。

在本文第二章中，我们将经典的欧拉多项式进行了推广，从原本的单统计量推广至四元统计量，分别计算了这个多项式在置换和错排上的分布，并且用数学归纳法和建立了一一映射两种方法证明了我们的等式。

在本文第三章中，基于推广的欧拉多项式，我们在置换和错排上分别定义了一种结构，圈下降置换和圈下降错排。在3.1和3.2节中，我们分别给出了这两种结构的基数的递归关系式，并且用一一映射的方法证明了这两个递归关系。在3.3节中，我们引入了一类特殊的完美匹配，即Callan完美匹配，我们证明了Callan完美匹配和圈下降置换之间存在一一映射。在3.4节中，我们了同样引入了一个结构L-S树，这个结构本来用于刻画二部图单纯复形的欧拉特征。在本文中，我们首先证明了L-S 树的基数和圈下降错排的基数相同且满足同样的递归关系，进而又给出了两个结构之间的一一映射。

关键字: 置换, 圈下降, 完美匹配, 欧拉多项式

THE RESEARCH ON THE CYCLE DESCENT STATISTICS OF PERMUTATIONS

ABSTRACT

The classical Eulerian polynomials was first proposed by Leonhard Euler in his report in 1749. It has been investigated widely in many areas of Mathematics. Specifically, the Eulerian polynomials has an interesting combinatorial form expressed by the the excedance of permutation group. Therefore, Some work was made in the excedance statistics of permutations. Some typical work was done by Foata and Brenti. However, when we look at this topic, we still find many interesting results and formulas that were unproved. In addition, we also notice that some of the formulas they give can be subdivided and show us a better combinatorial meaning. Therefore, in my thesis, we study the cycle descent statistic of permutations.

In Chapter 2, we generalize the classical Eulerian Polynomials and calculate its value on permutations and derangements respectively. Finally, we prove our formulas with both induction method and bijection method.

In Chapter 3, based on the generalized Eulerian Polynomials, we define two combinatorial objects on permutations and derangements, namely negative cycle descent permutations and negative cycle descent derangements. In section 3.1 and 3.2, we respectively present a recursion formula for these two objects and constructed bijection proofs for the formulas. In section 3.3, we introduce perfect matchings and we construct a bijection between a special type of perfect matchings, Callan perfect matching, and the negative cycle descent permutations. In section 3.4, we introduce the L-S trees, which is proved to be the reduced Euler characteristic for the simplicial complex of bipartite graphs. We first prove the L-S trees and negative cycle descent derangements have the same cardinality and recursion formula. Then, we construct a bijection between the two objects.

Keywords: Permutations, Cycle descents, Perfect matchings, Eulerian polynomials

Contents

Chapter 1 Introduction

1.1 Background

Let $\mathfrak{S}_n$ be the symmetric group of all permutations of $[n]$, where $[n] := \{1, 2, \ldots, n\}$. We write an element π in $\mathfrak{S}_n$ as $\pi = \pi(1)\pi(2)\cdots\pi(n)$. An *excedance* in π is an index i such that $\pi(i) > i$ and a *fixed point* in π is an index i such that $\pi(i) = i$. A fixed-point-free permutation is called a *derangement*. Denote by $\mathscr{D}_n$ the set of derangements of $[n]$. For any permutation $\pi \in \mathfrak{S}_n$, it is natural to consider the sequence $x, \pi(x), \pi^2(x), \ldots$ for each $x \in [n]$. Eventually, we must return to x. Thus, we have that $\pi^k(x) = x$ for some unique $k \geq 1$ and that the elements $x, \pi(x), \ldots, \pi^{k-1}(x)$ are distinct. The sequence $(x, \pi(x), \ldots, \pi^{k-1}(x))$ is called a cycle of π of length k. The cycles $(\pi^i(x), \pi^{i+1}(x), \ldots, \pi^{k-1}(x), x, \pi(x), \ldots, \pi^{i-1}(x))$ and $(x, \pi(x), \ldots, \pi^{k-1}(x))$ are considered equivalent. Every element of $[n]$ appears in a unique cycle of π, and π may be regarded as a disjoint union or product of its distinct cycles $C_1, \ldots, C_l$ and be written as $\pi = C_1 \cdots C_l$. As usual, let $\mathrm{exc}\,(\pi)$, $\mathrm{fix}\,(\pi)$ and $\mathrm{cyc}\,(\pi)$ denote the number of excedances, fixed points and cycles in π respectively. For example, the permutation $\pi = 3142765$ has the cycle decomposition $(1342)(57)(6)$, so $\mathrm{cyc}\,(\pi) = 3$, $\mathrm{exc}\,(\pi) = 3$ and $\mathrm{fix}\,(\pi) = 1$.

The Eulerian polynomials $A_n(x)$ are defined by

$$A_0(x) = 1, \quad A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} \quad \text{for } n \geq 1,$$

and have been extensively investigated. Foata and Schützenberger [6] introduced a q-analog of the Eulerian polynomials defined by

$$A_n(x; q) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} q^{\mathrm{cyc}\,(\pi)}.$$

Brenti [2, 3] further studied q-Eulerian polynomials and established the link with q-symmetric functions arising from plethysm. Brenti [3, Proposition 7.3] obtained the exponential generating function for $A_n(x; q)$:

$$1 + \sum_{n \geq 1} A_n(x; q) \frac{z^n}{n!} = \left(\frac{1 - x}{e^{z(x-1)} - x} \right)^q.$$

Remarkably, Brenti [3, Corollary 7.4] derived the following identity:

$$\sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cyc}\,(\pi)} = -(x-1)^{n-1}. \tag{1.1}$$

From then on, there is a large of literature devoted to various generalizations and refinements of the joint distribution of excedances and cycles (see [1, 5, 8, 14] for instance). For example, Ksavrelof and Zeng [8] constructed bijective proofs of Identity (1.1) and the following formula:

$$\sum_{\pi\in\mathscr{D}_n} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cyc}\,(\pi)} = -x-x^2-\cdots-x^{n-1}.$$

In particular, their bijection leads to a refinement of the above identity:

$$\sum_{\pi\in\mathscr{D}_{n,i}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cyc}\,(\pi)} = -x^{n-i},$$

where $\mathscr{D}_{n,i}$ is the set of derangements π of $[n]$ such that $\pi(n)=i$.

A *standard cycle decomposition* of $\pi\in\mathfrak{S}_n$ is defined by requiring that each cycle is written with its smallest element first, and the cycles are written in increasing order of their smallest element. A permutation is said to be cyclic if there is only one cycle in its cycle decomposition.

Definition 1.1. Let π be a permutation in $\mathfrak{S}_n$. Suppose that $(c_1, c_2, \ldots, c_i)$ is a cycle in standard cycle decomposition of π. We say that c_j is a *cycle descent* if $c_j > c_{j+1}$, where $1<j<i$. Denote by $CDES(\pi)$ the set of cycle descents of π and let $\mathrm{cdes}\,(\pi)=|CDES(\pi)|$ be the number of cycle descents of π.

For example, for $\pi=(1342)(57)(6)$, we have $CDES(\pi)=\{4\}$ and $\mathrm{cdes}\,(\pi)=1$. For $\pi\in\mathfrak{S}_n$, it is clear that $\mathrm{exc}\,(\pi)+\mathrm{cyc}\,(\pi)+\mathrm{cdes}\,(\pi)=n$. Thus

$$A_n(x;q) = q^n\sum_{\pi\in\mathfrak{S}_n}\left(\frac{x}{q}\right)^{\mathrm{exc}\,(\pi)}\left(\frac{1}{q}\right)^{\mathrm{cdes}\,(\pi)}.$$

Let $\mathfrak{S}_{n,i}$ be the set of permutations $\pi\in\mathfrak{S}_n$ with $\pi(i)=1$. For any $\pi\in\mathfrak{S}_n$, let π^{-1} denote the inverse of π, so $\pi^{-1}(1) = i$ if $\pi \in \mathfrak{S}_{n,i}$. For $n \geq 2$, we recently observed the following formulas:

$$\sum_{\pi \in \mathfrak{S}_{n,i}} (-1)^{\mathrm{cdes}\,(\pi)} t^{\pi^{-1}(1)} = \begin{cases} 2^{n-2}t & \text{if} \quad i = 1, \\ 0 & \text{if} \quad i = 2, \ldots, n-1, \\ 2^{n-2}t^n & \text{if} \quad i = n, \end{cases}$$

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{cdes}\,(\pi)} t^{\pi^{-1}(1)} = 2^{n-2}(t + t^n),$$

$$\sum_{\pi \in \mathscr{D}_n} x^{exc(\pi)} (-1)^{cdes(\pi)} t^{\pi^{-1}(1)} = \sum_{i=2}^{n} (-1)^{n-i} x^{i-1} t^i,$$

$$\sum_{\pi \in \mathscr{D}_n} (-1)^{cdes(\pi)} = \frac{1}{2}[1 - (-1)^{n-1}].$$

The above formulas can be easily proved by taking $x = 1$ in Theorems 1.2 and 1.3 of Chapter 2. Motivated by the these formulas, we shall study the cycle descent statistic of permutations.

1.2 Main Results of This Thesis

1.2.1 Cycle Descent Statistics on Permutations and Derangements

Consider the following enumerative polynomials

$$P_{n,i}(x, y, q, t) = \sum_{\pi \in \mathfrak{S}_{n,i}} x^{\mathrm{exc}\,(\pi)} y^{\mathrm{cdes}\,(\pi)} q^{\mathrm{fix}\,(\pi)} t^{\pi^{-1}(1)}.$$

It is remarkable that the polynomials $P_{n,i}(x, -1, 1, t)$ and $P_{n,i}(x, -1, 0, t)$ have simple closed formulas. We state them as the first main result of this thesis.

Theorem 1.2. *(Theorem 2.2 of Chapter* 2*)*

For $n \geq 2$*, we have*

$$P_{n,i}(x, -1, 1, t) = \begin{cases} t(1+x)^{n-2} & \textit{if} \quad i = 1, \\ 0 & \textit{if} \quad i = 2, \ldots, n-1, \\ t^n x(1+x)^{n-2} & \textit{if} \quad i = n, \end{cases}$$

We give two proofs of Theorem 1.2 in Chapter 2. The first proof is an inductive proof. Note that

$$P_{n,i}(x, -1, 1, t) = \sum_{\pi \in \mathfrak{S}_{n,i}} x^{\mathrm{exc}\,(\pi)} (-1)^{\mathrm{cdes}\,(\pi)} t^{\pi^{-1}(1)}.$$

The second proof is based on an involution which is established on $\mathfrak{S}_{n,i}$.

Theorem 1.3. *(Theorem 2.4 of Chapter 2)*

For $n \geq 2$, $i \leq n$, we have

$$P_{n,i}(x, -1, 0, t) = (-1)^{n-i} x^{i-1} t^i.$$

We also give two proofs of Theorem 1.3 in Chapter 2. The first proof is an inductive proof. Note that

$$P_{n,i}(x, -1, 0, t) = \sum_{\pi \in \mathfrak{S}_{n,i}} x^{\mathrm{exc}\,(\pi)} (-1)^{\mathrm{cdes}\,(\pi)} 0^{fix(\pi)} t^{\pi^{-1}(1)} = \sum_{\pi \in \mathscr{D}_{n,i}} x^{\mathrm{exc}\,(\pi)} (-1)^{\mathrm{cdes}\,(\pi)} t^{\pi^{-1}(1)}.$$

The second proof is based on an involution which is established on $\mathscr{D}_{n,i}$. Moreover, if taking $t = 1$ in the above two theorems and applying the formula exc(π) + cyc(π) + cdes(π) = n, it will lead to Ksavrelof and Zeng's results in [8].

1.2.2 Negative Cycle Descent Permutations

A signed permutation (π, ϕ) of $[n]$ is a permutation $\pi \in \mathfrak{S}_n$ together with a map $\phi : [n] \mapsto \{+1, -1\}$ and we call $\phi(i)$ the sign of i. For simplicity, we indicate the sign of $\pi(i)$ by writing $\pi(i)^+$ or $\pi(i)^-$. The group, which consists of all the signed permutations of $[n]$ with composition as the group operation, is called the signed permutation group of order n. This group, also known as the Weyl group of type B or as the hyperoctahedral group, is fundamental objects in today’s mathematics.

Let (π, ϕ) be a signed permutation. Let $NEG(\pi, \phi)$ be the set of numbers $\pi(i)$ with the sign -1, i.e.

$$NEG(\pi, \phi) = \{\pi(i) \mid \phi(\pi(i)) = -1\},$$

and let $neg(\pi, \phi) = |NEG(\pi, \phi)|$.

Definition 1.4. A negative cycle descent permutation (π, ϕ) of $[n]$ is a signed permutation (π, ϕ) such that $NEG(\pi, \phi) \subseteq CDES(\pi)$.

Let

$$b_n(y,q)=\sum_{i=1}^{n}P_{n,i}(1,y,q,1)=\sum_{\pi\in\mathfrak{S}_n}y^{\mathrm{cdes}\,(\pi)}q^{\mathrm{fix}\,(\pi)}.$$

It is easy to verify that $b_n(2,1)$ is the number of negative cycle descent permutations of $[n]$ since

$$b_n(2,1)=\sum_{\pi\in\mathfrak{S}_n}2^{\mathrm{cdes}\,(\pi)}$$

and $b_n(2,0)$ is the number of negative cycle descent derangements of $[n]$ since

$$b_n(2,0)=\sum_{\pi\in\mathfrak{S}_n}2^{\mathrm{cdes}\,(\pi)}0^{\mathrm{fix}\,(\pi)}=\sum_{\pi\in\mathscr{D}_n}2^{\mathrm{cdes}\,(\pi)}.$$

We present the second main result of this thesis as follows.

Theorem 1.5. *(Theorem 3.1 of Chapter* 3*)*

For $n\geq 1$*, we have*

$$b_{n+1}(y,1)=b_n(y,1)+\sum_{i=1}^{n}b_i(y,1)\binom{n}{i-1}(y-1)^{n-i}$$

with the initial condition $b_1(y,1)=1$*.*

Theorem 1.6. *(Theorem 3.2 of Chapter* 3*)*

For $n\geq 1$*, we have*

$$b_{n+1}(y,0)=\sum_{i=0}^{n-1}\binom{n}{i}\left[b_{i+1}(y,0)+b_i(y,0)\right](y-1)^{n-i-1}$$

with initial conditions $b_0(y,0)=1, b_1(y,0)=0$*.*

By taking $y=2$ in the identity of Theorem 1.5, we could obtain Klazar's recurrence for $w_{12}(n)$ (see [7, Eq. (39)] for details), which can be written as follows:

$$b_{n+1}(2,1)=b_n(2,1)+\sum_{i=1}^{n}b_{n+1-i}(2,1)\binom{n}{i}.$$

In [4, 7, 11], the sets of some combinatorial objects, which have cardinality $b_n(2,1)$, were studied. We list some of them as follows:

(i) The set of drawings of rooted plane trees with $n+1$ vertices (see [7]);

(ii) The set of Klazar trees with $n+1$ vertices (see [4]);

(iii) The set of perfect matchings on the set $[2n]$ in which no even number is matched to a larger odd number (see [4]).

(iv) The set of ordered partitions of $[n]$ all of whose left-to-right minima occur at odd locations (see [11]).

1.2.3 Negative Cycle Descent Permutations and Callan Perfect Matching

Now we begin to introduce the concept of perfect matchings. Let $\mathbb{P}_A = A \times \{0,1\}$, where $A = \{i_1, \ldots, i_k\}$ is a finite set of positive integers with $i_1 < i_2 < \cdots < i_k$. When $A = [n]$, we write $\mathbb{P}_A$ as $\mathbb{P}_n$. A perfect matching is a partition of $\mathbb{P}_A$ into 2-element subsets or matches. For any match $\{(i,x),(j,y)\}$ in a perfect matching, we say that (i,x) is the *partner* of (j,y). For convenience, we represent a perfect matching as a dot diagram with vertices arranged in two rows.

Example 1.7. *We give a dot diagram of a perfect matching M of $\mathbb{P}_8$ as follows:*

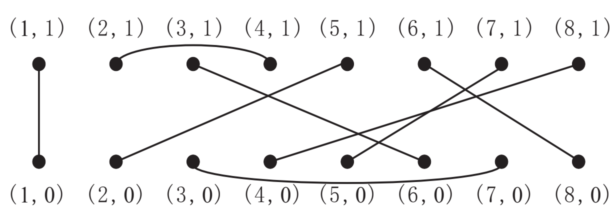


Fig.1. A perfect matching M of $\mathbb{P}_8$

Thus, for any perfect matching M of $\mathbb{P}_A$, we say that $\mathbb{P}_A$ is the vertex set of M and every match is an edge of M. We use $V(M)$ and $E(M)$ to denote vertices set and edges set in M respectively. Moreover, an edge is called an *arc* if it joins two dots in the same row; otherwise, this edge is called a *line*. For any line $\{(i,0),(j,1)\}$, it is said to be a *upline* if $i < j$, a *downline* if $i > j$ and a *vertical* line if $i = j$. For any perfect matching M, let $\mathrm{arc}\,(M), \mathrm{down}\,(M)$ and $\mathrm{ver}\,(M)$ be the numbers of arc, down lines and vertical lines in M respectively.

Example 1.8. *In the perfect matching of Example 1.7, the edge $\{(1,1),(1,0)\}$ is a vertical line, the edges $\{(3,1),(6,0)\}$ and $\{(6,1),(8,0)\}$ are two downlines, the edges $\{(5,1),(2,0)\}$, $\{(7,1),(5,0)\}$*

and $\{(8,1),(4,0)\}$ *are three uplines, the edges* $\{(2,1),(4,1)\}$ *and* $\{(3,0),(7,0)\}$ *are two arcs; finally,* $arc(M)=2, down(M)=2, ver(M)=1$.

Definition 1.9. A perfect matching M of $\mathbb{P}_n$ is a Callan perfect matching if M has no uplines.

Example 1.10. *We give a dot diagram of a Callan perfect matching* M *of* $\mathbb{P}_8$ *as follows:*

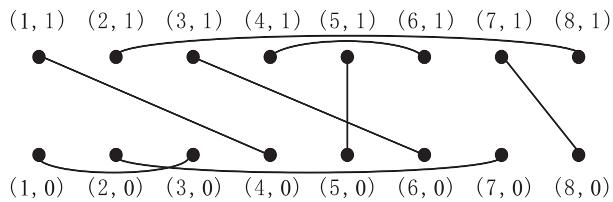


Fig.2. A Callan perfect matching M *of* $\mathbb{P}_8$

Let m_n be the number of Callan perfect matchings of $\mathbb{P}_n$. Callan [4] proved that m_n satisfies the recurrence (1.2.2). So the number of negative cycle descent permutations of $[n]$ equals to the number of Callan perfect matching of $\mathbb{P}_n$.

Let M be a perfect matching of $\mathbb{P}_n$. We say that M' is a *sub-perfect matching* of M if M' is a perfect matching such that $V(M') \subseteq V(M)$ and $E(M') \subseteq E(M)$. For any $V \subseteq [n]$, if there is a sub-perfect matching M' of M with $V(M') = V \times \{0,1\}$, then M' is said to be the sub-perfect matching induced by V and is denoted by $M[V]$.

Denote by $\mathcal{G}(M)$ a graph which is obtained from M by identifying each two vertices $(i,0)$ and $(i,1)$ as a new vertex i for any $i \in [n]$. It is easy to see that the graph $\mathcal{G}(M)$ is the union of some disjoint cycles. For a cycle C in $\mathcal{G}(M)$, suppose C has the vertices set V. Note that there is a sub-perfect matching of M induced by V. We say that $M[V]$ is a connected component of M. Let $\mathrm{com}\,(M)$ be the number of connected components in a perfect matching M. If a perfect matching M has exactly one connected component, i.e., $\mathrm{com}\,(M) = 1$, then we say that M is a connected perfect matching.

Example 1.11. *For the perfect matching* M *of Example 1.10, we draw the graph* $\mathcal{G}(M)$ *as follows:*

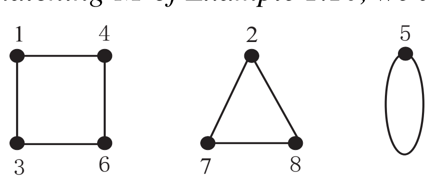


Fig.3. A graph $\mathcal{G}(M)$*.*

So we have $\operatorname{com}(M) = 3$.

We discover a connection between negative cycle descent permutations and Callan perfect matchings and present the third main result of this thesis as follows.

Theorem 1.12. *(Theorem 3.5 of Chapter 3) There is a bijection Γ_n between the set of negative cycle descent permutations of $[n]$ and the set of Callan perfect matchings of $\mathbb{P}_n$. Moreover, for any negative cycle descent permutation (π, ϕ) of $[n]$, we have*

$$\operatorname{com}(\Gamma_n(\pi, \phi)) = \operatorname{cyc}(\pi),\ \operatorname{ver}(\Gamma_n(\pi, \phi)) = \operatorname{fix}(\pi),$$

and

$$\operatorname{down}(\Gamma_n(\pi, \phi)) = \begin{cases} \operatorname{neg}(\pi, \phi) & \text{if } (1,1) \text{ and its partner are in the same row,} \\ \operatorname{neg}(\pi, \phi) + 1 & \text{otherwise.} \end{cases}$$

Let $\Gamma|_{\mathscr{D}_n}$ denote the restriction of Γ_n on the set of negative cycle descent derangements of $[n]$. So the following corollary is immediate.

Corollary 1.13. *(Corollary 3.7 of Chapter 3)*

$\Gamma|_{\mathscr{D}_n}$ *is a bijection between the set of negative cycle descent derangements of $[n]$ and the set of Callan perfect matchings of $\mathbb{P}_n$ which have no vertical lines.*

1.2.4 Negative Cycle Descent Derangements and L-S Trees

Let T be a tree on vertex set $\{0, 1, \ldots, n\}$ rooted at 0. For any vertex $i \neq 0$, there is a unique path connecting i and the root 0. If a vertex j is the first vertex lying on the path, we say that j is the parent of i, i is a child of j. Define the height of i to be the number of edges in this path. The height of the root 0 is zero. For any $k \geq 0$, let $V_k(T)$ be the set of vertices in T whose height is k and

$$m(T) = \max\{k \mid V_k(T) \neq \emptyset, k \geq 0\}.$$

For $k = 1, 2, \ldots, m(T)$, define

$$a_k(T) = \min V_k(T).$$

Definition 1.14. For any $n \geq 1$, let $\mathcal{T}_n$ be the set of all trees T rooted at 0, on vertex set $\{0, 1, \ldots, n\}$ which satisfies the following three conditions:

(i) for $k = 1, 2, \ldots, m(T) - 1$, the only elemmaent of $V_k(T)$ which has children is $a_k(T)$;

(ii) for $k = 1, 2, \ldots, m(T) - 1$, there is an elemmaent of $\bigcup_{j \geq k} V_j(T)$ which is larger than $a_k(T)$;

(iii) $|V_{m(T)}(T)| \geq 2$.

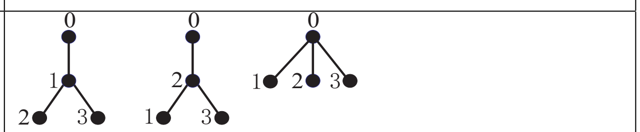
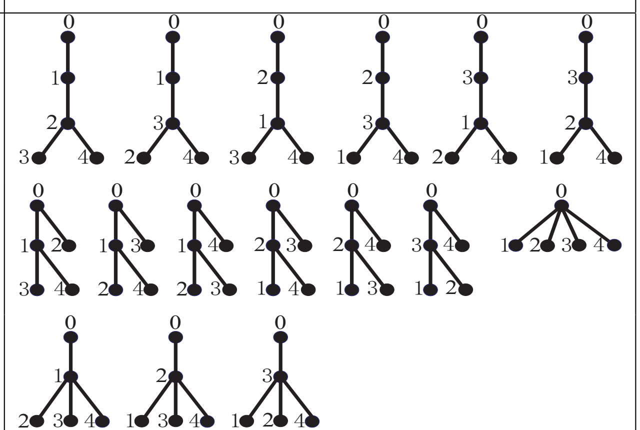
Let $t_0 = 1$ and $t_n = |\mathcal{T}_n|$ for any $n \geq 1$. Linusson and Shareshian [9] proved that t_n is the reduced Euler characteristic for the simplicial complex of bipartite graphs on n+1 vertices. So we call them L-S trees. Define a generating function

$$f(x) = 1 + \sum_{n \geq 1} t_n \frac{x^n}{n!}.$$

By Exercise 5.5 of [12], it is easy to obtain that

$$f(x) = \sqrt{\frac{1}{2e^x - e^{2x}}}.$$

Example 1.15. *In the following table, we draw all the L-S trees with* $n + 1$ *vertices for* $n = 2, 3, 4$.

$\mathcal{T}_2$:	
$\mathcal{T}_3$:	
$\mathcal{T}_4$:	

Recall that $b_n(2, 0)$ is the number of negative cycle descent derangements of $\{1, 2, \ldots, n\}$. We present the fourth main result of this thesis as follows.

Theorem 1.16. *(Theorem 3.8 of Chapter* 3*)*

Let $t_0 = 1$ *and* $t_1 = 0$*. Then for* $n \geq 1$*, we have*

$$t_{n+1} = \sum_{i=0}^{n-1} \binom{n}{i} (t_{i+1} + t_i)$$

and

$$t_n = b_n(2, 0)$$

Theorem 1.17. *(Theorem 3.11 of Chapter* 3*)*

For $n \geq 2$*, there is a bijection* Θ_n *between the set of negative cycle descent derangements of* $[n]$ *and* $\mathcal{T}_n$*.*

Chapter 2 Cycle Descent Statistics

In this chapter, we will study cycle descent statistics on $\mathfrak{S}_n$ and $\mathscr{D}_n$.

2.1 Cycle Descent Statistics on Permutations

Suppose that $\pi = \pi(i_1)\dots\pi(i_k)$ is a permutation on the set $\{i_1,\dots,i_k\}$ of positive integers with $i_1 < i_2 < \cdots < i_k$. Throughout this paper, we always let

$$\mathrm{red}\,(\pi) := \mathrm{red}\,(\pi(i_1))\dots\mathrm{red}\,(\pi(i_k)) \in \mathfrak{S}_k,$$

where red is a increasing map from $\{i_1,\dots,i_k\}$ to $\{1,2,\dots,k\}$ defined by $\mathrm{red}\,(i_j) = j$ for any $j = 1,2,\dots,k$.

Let $P_n(x,y,1,1) = \sum\limits_{i=1}^{n} P_{n,i}(x,y,1,1)$. We give a recurrence for $P_{n,i}(x,y,1,1)$ in the following lemma.

Lemma 2.1. *For any $n \geq 2$, we have*

$$P_{n+1,i}(x,y,1,1) = \begin{cases} P_n(x,y,1,1) & \textit{if} \quad i=1, \\ xP_{n-1}(x,y,1,1) + x\sum\limits_{j=2}^{i-1} P_{n,j}(x,y,1,1) + y\sum\limits_{j=i}^{n} P_{n,j}(x,y,1,1) & \textit{if} \quad i=2,\dots,n+1, \end{cases}$$

with initial conditions

$$P_{1,1}(x,y,1,1) = 1, P_{2,1}(x,y,1,1) = 1, P_{2,2}(x,y,1,1) = x.$$

Proof. For any $\pi = \pi(1)\pi(2)\dots\pi(n+1) \in \mathfrak{S}_{n+1,1}$, we have $\pi(1) = 1$. Let $\tilde{\pi} = \pi(2)\dots\pi(n+1)$. Then $\tilde{\pi}$ is a permutation on the set $\{2,3,\dots,n\}$ and $\mathrm{red}\,(\tilde{\pi}) \in \mathfrak{S}_n$. Obviously,

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tilde{\pi})) \text{ and } \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tilde{\pi})).$$

So we have $P_{n+1,1}(x,y,1,1) = P_n(x,y,1,1)$.

For any $i \geq 2$, let $\pi = \pi(1)\pi(2)\dots\pi(n+1) \in \mathfrak{S}_{n+1,i}$. Let $\sigma = (1,c_1,c_2,\dots,c_m)$ be the cycle in the standard cycle decomposition of π which contains the entry 1. So π can be split into the cycle σ and a permutation τ on the set $\{1, 2, \ldots, n+1\} \setminus \{1, c_1, \ldots, c_m\}$, i.e., $\pi = \sigma \cdot \tau$. Clearly, $m \geq 1$, $i \geq 2$ and $c_m = i$ since $\pi \in \mathfrak{S}_{n+1,i}$. We distinguish between the following two cases:

Case 1. $m = 1$.

Deleting the cycle $(1, c_1) = (1, i)$ from the standard cycle decomposition of π, we obtain the permutation

$$\tau = \pi(2) \ldots \pi(i-1)\pi(i+1) \ldots \pi(n+1)$$

which is defined on the set $\{1, 2, \ldots, n+1\} \setminus \{1, i\}$. Note that $\mathrm{red}\,(\tau) \in \mathfrak{S}_{n-1}$ and

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tau)) + 1, \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tau)).$$

This provides the term

$$xP_{n-1}(x, y, 1, 1).$$

Case 2. $m \geq 2$.

Suppose that $c_{m-1} = j$ for some $2 \leq j \leq n+1$. Deleting the number $c_m = i$ from the standard cycle decomposition of π, we obtain a permutation

$$\tilde{\pi} = (1, c_1, \ldots, c_{m-1}) \cdot \tau$$

which is defined on the set $\{1, \ldots, i-1, i+1, \ldots, n+1\}$. Hence $\mathrm{red}\,(\tilde{\pi}) \in \mathfrak{S}_n$. Moreover, if $c_{m-1} = j \leq i-1$, then

$$\mathrm{red}\,(\tilde{\pi}) \in \mathfrak{S}_{n,j}, \mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tilde{\pi})) + 1, \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tilde{\pi})).$$

This provides the term

$$x \sum_{j=2}^{i-1} P_{n,j}(x, y, 1, 1).$$

If $c_{m-1} = j \geq i+1$, then

$$\mathrm{red}\,(\tilde{\pi}) \in \mathfrak{S}_{n,j-1}, \mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tilde{\pi})), \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tilde{\pi})) + 1.$$

This provides the term

$$y \sum_{j=i}^{n} P_{n,j}(x, y, 1, 1).$$

In conclusion, for any $i \geq 2$ we have

$$P_{n+1,i}(x,y,1,1) = xP_{n-1}(x,y,1,1) + x\sum_{j=2}^{i-1} P_{n,j}(x,y,1,1) + y\sum_{j=i}^{n} P_{n,j}(x,y,1,1).$$

□

Theorem 2.2. *For $n \geq 2$, we have*

$$P_{n,i}(x,-1,1,t) = \begin{cases} t(1+x)^{n-2} & \textit{if} \quad i=1, \\ 0 & \textit{if} \quad i=2,\ldots,n-1, \\ t^n x(1+x)^{n-2} & \textit{if} \quad i=n, \end{cases} \tag{2.1}$$

Proof. Note that

$$\sum_{\pi\in\mathfrak{S}_{n,i}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} t^{\pi^{-1}(1)} = t^i \sum_{\pi\in\mathfrak{S}_{n,i}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = t^i P_{n,i}(x,-1,1,1).$$

In order to prove the identity (2.1) in Theorem 2.2, it is sufficient to show that

$$P_{n,i}(x,-1,1,1) = \begin{cases} (1+x)^{n-2} & \text{if} \quad i=1, \\ 0 & \text{if} \quad i=2,\ldots,n-1, \\ x(1+x)^{n-2}, & \text{if} \quad i=n. \end{cases} \tag{2.2}$$

(First proof of Identity (2.2)) We fist prove Identity (2.2) by induction on the integer n.

It is easy to verify that

$$P_{2,1}(x,-1,1,1) = 1, P_{2,2}(x,-1,1,1) = x.$$

Assume that the explicit formula (2.2) holds for any $2 \leq k \leq n$. By Lemma 2.1, we have

$$\begin{aligned} P_{n+1,1}(x,-1,1,1) &= P_n(x,-1,1,1) \\ &= P_{n,1}(x,-1,1,1) + P_{n,n}(x,-1,1,1) \\ &= (1+x)^{n-2} + x(1+x)^{n-2} = (1+x)^{n-1}, \end{aligned}$$

$$\begin{aligned} P_{n+1,n+1}(x,-1,1,1) &= xP_{n-1}(x,-1,1,1) + x\sum_{j=2}^{n} P_{n,j}(x,-1,1,1) \\ &= xP_{n-1,1}(x,-1,,1,1) + xP_{n-1,n-1}(x,-1,1,1) + xP_{n,n}(x,-1,1,1) \\ &= x(1+x)^{n-2} + x^2(1+x)^{n-2} = x(1+x)^{n-1} \end{aligned}$$

and

$$\begin{aligned} P_{n+1,i}(x,-1,1,1) &= xP_{n-1}(x,-1,1,1) + x\sum_{j=2}^{i-1} P_{n,j}(x,-1,1,1) - \sum_{j=i}^{n} P_{n,j}(x,-1,1,1) \\ &= xP_{n-1,1}(x,-1,1,1) + xP_{n-1,n-1}(x,-1,1,1) - P_{n,n}(x,-1,1,1) \\ &= x(1+x)^{n-2} - P_{n,n}(x,-1,1,1) = 0 \end{aligned}$$

for any $2 \le i \le n$.

(Second proof of Identity (2.2)) Now we give a bijective proof of Identity (2.2) by establishing an involution $\psi_{n,i}$ on $\mathfrak{S}_{n,i}$.

For any $\pi \in \mathfrak{S}_n$, suppose that $\pi = C_1 \dots C_k$ is the standard cycle decomposition of π. Let

$$\hat{\pi} = a_1 a_2 \dots a_n$$

be the permutation obtained from π by erasing the parentheses in its standard cycle decomposition. Furthermore, for any $i = 1, 2, \dots, n-1$, the number a_i is said to be a value-descent of π if $a_i > a_{i+1}$ in the sequence $\hat{\pi}$, and let q_π be the *last value-descent* which appears in the sequence $\hat{\pi}$. For example, the permutation $\pi = 1472365$ in $\mathfrak{S}_7$ has the standard cycle decomposition $(1)(24)(375)(6)$, so $\hat{\pi} = 1243756$, it has exactly two value-descents 4 and 7, and $q_\pi = 7$. The process of erasing the parentheses from the standard cycle decomposition of a permutation is well-known bijection of Foata and Schutzenberger, the "transformation fundamental", see also [6].

We define a map $\Phi : \mathfrak{S}_n \mapsto \mathfrak{S}_n$ as follows:

For any $\pi \in \mathfrak{S}_n$, if q_π is the last element of a cycle C_i for some i, then let $\Phi(\pi)$ be the permutation obtained from π by erasing the right and left parentheses ")(" after the number q_π in the standard cycle decomposition of π; otherwise, let $\Phi(\pi)$ be the permutation obtained from π by inserting a right parentheses ")" and a left parentheses "(" after the number q_π in the standard cycle decomposition of π. For example, if $\pi = (1)(24)(375)(6)$, then $\hat{\pi} = 1243756$ and $q_\pi = 7$, and so $\Phi(\pi) = (1)(24)(37)(5)(6)$. If $\sigma = (1)(24)(37)(5)(6)$, then $\Phi(\sigma) = (1)(24)(375)(6)$. Clearly, we have

$$\hat{\pi} = \widehat{\Phi(\pi)},\ q_\pi = q_{\Phi(\pi)},\ \mathrm{exc}\,(\pi) = \mathrm{exc}\,(\Phi(\pi)),\ \mathrm{cdes}\,(\pi) - \mathrm{cdes}\,(\Phi(\pi)) = \pm 1.$$

Denote by $\Omega_{n,1}$ the set of the permutations $\pi \in \mathfrak{S}_{n,1}$ such that $\hat{\pi} = 123, \ldots n$. For any $\pi \in \Omega_{n,1}$, suppose that

$$\pi = (1)C_1C_2 \ldots C_{k-1}C_k$$

is the standard cycle decomposition of π. Let i_s be the largest number in the cycle C_s for every $s = 1, 2, \ldots, k-1$. Then the set $\{i_1, i_2, \ldots, i_{k-1}\}$ is a subset of the set $\{2, 3, \ldots, n-1\}$ and $i_1 < i_2 < \ldots < i_{k-1}$.

Conversely, suppose that $\{i_1, i_2, \ldots, i_{k-1}\}$ is a subset of the set $\{2, 3, \ldots, n-1\}$ and $i_1 < i_2 < \ldots < i_{k-1}$. Let

$$\pi = (1)(2, 3, \ldots, i_1)(i_1{+}1, i_1{+}2, \ldots, i_2) \ldots (i_{k-2}{+}1, i_{k-2}{+}2, \ldots, i_{k-1})(i_{k-1}{+}1, i_{k-1}{+}2, \ldots, n).$$

We have $\pi \in \Omega_{n,1}$ and $exc(\pi) = n - k - 1$. Thus, the weight of $\Omega_{n,1}$ is

$$\sum_{\pi\in\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = \sum_{\pi\in\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)} = \sum_{k=1}^{n-1}\binom{n-2}{k-1}x^{n-k-1} = \sum_{k=0}^{n-2}\binom{n-2}{k}x^{n-k-2} = (x{+}1)^{n-2}.$$

For any permutation $\pi \in \mathfrak{S}_{n,1} \setminus \Omega_{n,1}$, we have $\Phi(\pi) \in \mathfrak{S}_{n,1} \setminus \Omega_{n,1}$. So for any $\pi \in \mathfrak{S}_{n,1}$, let

$$\psi_{n,1}(\pi) = \begin{cases} \Phi(\pi) & \text{if} \quad \pi \in \mathfrak{S}_{n,1} \setminus \Omega_{n,1}, \\ \pi & \text{if} \quad \pi \in \Omega_{n,1}. \end{cases}$$

Note that

$$\begin{aligned}
\sum_{\pi\in\mathfrak{S}_{n,1}\setminus\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} &= -\sum_{\pi\in\mathfrak{S}_{n,1}\setminus\Omega_{n,1}} x^{\mathrm{exc}\,(\Phi(\pi))}(-1)^{\mathrm{cdes}\,(\Phi(\pi))} \\
&= -\sum_{\pi'\in\mathfrak{S}_{n,1}\setminus\Omega_{n,1},\pi=\Phi(\pi')} x^{\mathrm{exc}\,(\Phi(\pi))}(-1)^{\mathrm{cdes}\,(\Phi(\pi))} \\
&= -\sum_{\pi'\in\mathfrak{S}_{n,1}\setminus\Omega_{n,1}} x^{\mathrm{exc}\,(\pi')}(-1)^{\mathrm{cdes}\,(\pi')}.
\end{aligned}$$

This implies that

$$\sum_{\pi\in\mathfrak{S}_{n,1}\setminus\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = 0.$$

Hence,

$$\begin{aligned}
\sum_{\pi\in\mathfrak{S}_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} &= \sum_{\pi\in\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} + \sum_{\pi\in\mathfrak{S}_{n,1}\setminus\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} \\
&= \sum_{\pi\in\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = \sum_{\pi\in\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)} = (1+x)^{n-2}.
\end{aligned}$$

For example, we list all $\pi \in \mathfrak{S}_{4,1}$ and $\psi_{4,1}(\pi)$ in Table 1.

$\pi \in \mathfrak{S}_{4,1}$	$x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)}$	$\hat{\pi}$	q_π	$\psi_{4,1}(\pi)$
$(1)(2)(3)(4)$	1	1234		$(1)(2)(3)(4)$
$(1)(23)(4)$	x	1234		$(1)(23)(4)$
$(1)(2)(34)$	x	1234		$(1)(2)(34)$
$(1)(234)$	x^2	1234		$(1)(234)$
$(1)(24)(3)$	x	1243	4	$(1)(243)$
$(1)(243)$	$-x$	1243	4	$(1)(24)(3)$

Table.1. Involution $\psi_{4,1}$

For $2 \leq i \leq n$, denote by $\mathcal{A}_{n,i}$ the set of permutations $\pi \in \mathfrak{S}_{n,i}$ such that the number q_π isn't in the first cycle in the standard cycle decomposition of π. Let $\mathcal{A}^*_{n,i} = \mathfrak{S}_{n,i} \setminus \mathcal{A}_{n,i}$ for short. For any $\pi \in \mathcal{A}^*_{n,i}$, let $\pi = C_1 \ldots C_k$ be the standard cycle decomposition of π, suppose the length of the cycle C_1 is $l+1$ and

$$\hat{\pi} = 1, \ldots, i, a_1, \ldots, a_{n-l-1}.$$

Then we have

$$a_1 < a_2 < \ldots < a_{n-l-1}$$

since q_π is an element in the cycle C_1.

Now suppose that $C_1 = (1, c_{11}, \ldots, c_{1l})$. Let Q be the set of indices $j \in \{1, 2, \ldots, l\}$ such that c_{1j} is not the largest number in the set $\{1, 2, \ldots, n\} \setminus \{c_{1,j+1}, \ldots, c_{1l}\}$, i.e.,

$$Q = \{j \mid 1 \leq j \leq l \text{ and } c_{1j} < \max\{1, 2, \ldots, n\} \setminus \{c_{1,j+1}, \ldots, c_{1l}\}\}.$$

Let $\Omega_{n,i}$ be the set of permutations $\pi \in \mathcal{A}^*_{n,i}$ such that $Q = \emptyset$. For any $i = 2, \ldots, n-1$, we have $l \in Q$ since $i < n$, and so $\Omega_{n,i} = \emptyset$. Moreover, $\pi \in \Omega_{n,n}$ if and only if

$$\hat{\pi} = 1, k, k+1, \ldots n-1, n, 2, 3, \ldots, k-2, k-1$$

for some $k > 1$.

We define a map Ψ from $\mathcal{A}^*_{n,i} \setminus \Omega_{n,i}$ to itself as follows:

For any $\pi \in \mathcal{A}^*_{n,i} \setminus \Omega_{n,i}$, let $m = m_\pi = \min Q$ since $Q \neq \emptyset$. If $m = 1$, then there are at least two cycles in the standard cycle decomposition of π. If $m \geq 2$, then we have

$$c_{11} < \cdots < c_{1,m-1} \text{ and } c_{1,m-1} > c_{1m}.$$

Furthermore, we distinguish between the following two cases:

Case 1. $2 \leq m \leq l$.

Let

$$\Psi(\pi) = (1, c_{1m}, \ldots, c_{1l}) \cdot C_2 \ldots C_k \cdot (c_{11}, \ldots, c_{1,m-1}).$$

Then $\Psi(\pi)$ has at least two cycles, $m_{\Phi(\pi)} = 1$ since $c_{1m} < c_{1,m-1}$, and so $\Psi(\pi) \in \mathcal{A}^*_{n,i} \setminus \Omega_{n,i}$. Moreover, we have $\operatorname{exc}(\pi) = \operatorname{exc}(\Psi(\pi))$ and $\operatorname{cdes}(\pi) = \operatorname{cdes}(\Psi(\pi)) + 1$

Case 2. $m = 1$.

Suppose that $C_1 = (1, c_{11}, \ldots, c_{1l})$ and $C_k = (c_{k1}, \ldots, c_{ks})$ are the first cycle and the last cycle in the standard cycle decomposition of π respectively, where s is the length of the cycle C_k. Let

$$\Psi(\pi) = (1, c_{k1}, \ldots, c_{ks}, c_{11}, \ldots, c_{1l}) \cdot C_2 \ldots C_{k-1}.$$

Then

$$m_{\Psi(\pi)} = s + 1 \geq 2,$$

and so $\Psi(\pi) \in \mathcal{A}^*_{n,i} \setminus \Omega_{n,i}$. Moreover, we have $\operatorname{exc}(\pi) = \operatorname{exc}(\Psi(\pi))$ and $\operatorname{cdes}(\pi) = \operatorname{cdes}(\Psi(\pi)) - 1$

When $2 \leq i \leq n-1$, for any $\pi \in \mathfrak{S}_{n,i}$, let

$$\psi_{n,i}(\pi) = \begin{cases} \Phi(\pi) & \text{if} \quad \pi \in \mathcal{A}_{n,i} \\ \Psi(\pi) & \text{if} \quad \pi \in \mathfrak{S}_{n,i} \setminus \mathcal{A}_{n,i} \end{cases} .$$

For example, we list all $\pi \in \mathfrak{S}_{4,2}$ and $\psi_{4,2}(\pi)$ in Table 2, and $\pi \in \mathfrak{S}_{4,3}$ and $\psi_{4,3}(\pi)$ in Table 3.

$\pi \in \mathfrak{S}_{4,2}$	$x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)}$	$\hat{\pi}$	q_π	m_π	$\psi_{4,2}(\pi)$
$(12)(3)(4)$	x	1234		1	$(142)(3)$
$(142)(3)$	$-x$	1423	4	2	$(12)(3)(4)$
$(12)(34)$	x^2	1234		1	(1342)
(1342)	$-x^2$	1342	4	3	$(12)(34)$
(1432)	x	1432	3	2	$(132)(4)$
$(132)(4)$	$-x$	1324	3	1	(1432)

Table.2. Involution $\psi_{4,2}$

$\pi \in \mathfrak{S}_{4,3}$	$x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)}$	$\hat{\pi}$	q_π	m_π	$\psi_{4,3}(\pi)$
$(13)(2)(4)$	x	1324	3	1	$(143)(2)$
$(143)(2)$	$-x$	1432	3	2	$(13)(2)(4)$
$(13)(24)$	x^2	1324	3	1	(1243)
(1243)	$-x^2$	1243	4	3	$(13)(24)$
(1423)	$-x^2$	1423	4	2	$(123)(4)$
$(123)(4)$	x^2	1234		1	(1423)

Table.3. Involution $\psi_{4,3}$

Hence,

$$\sum_{\pi\in\mathfrak{S}_{n,i}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = \sum_{\pi\in\mathcal{A}_{n,i}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} + \sum_{\pi\in\mathfrak{S}_{n,i}\setminus\mathcal{A}_{n,i}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = 0.$$

When $i = n$, we claim that the weight of $\Omega_{n,n}$ is $x(1+x)^{n-2}$. For any $\pi \in \Omega_{n,n}$, suppose that

$$\pi = (1, c_{11}, c_{12}, \ldots, c_{1s})C_1C_2 \ldots C_k$$

is the standard cycle decomposition of π, where $c_{1s} = n$. Let

$$\pi' = (1)C_1C_2 \ldots C_k(c_{11}, c_{12}, \ldots, c_{1s}).$$

Then $\pi' \in \Omega_{n,1}$ and $exc(\pi) = exc(\pi') + 1$. So, the weight of $\Omega_{n,n}$ is

$$\sum_{\pi\in\Omega_{n,n}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = \sum_{\pi\in\Omega_{n,n}} x^{\mathrm{exc}\,(\pi)} = \sum_{\pi'\in\Omega_{n,1}} x^{exc(\pi')+1} = x(1+x)^{n-2}.$$

For any $\pi \in \mathfrak{S}_{n,n}$, let

$$\psi_{n,n}(\pi) = \begin{cases} \Phi(\pi) & \text{if} \quad \pi \in \mathcal{A}_{n,n} \\ \Psi(\pi) & \text{if} \quad \pi \in \mathfrak{S}_{n,n} \setminus (\mathcal{A}_{n,n} \cup \Omega_{n,n}) \\ \pi & \text{if} \quad \pi \in \Omega_{n,n} \end{cases}.$$

For example, we list all $\pi \in \mathfrak{S}_{4,4}$ and $\psi_{4,4}(\pi)$ in Table 4.

π	$x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)}$	$\hat{\pi}$	q_π	m_π	$\psi_{4,4}(\pi)$
$(14)(2)(3)$	x	1423	4		$(14)(2)(3)$
$(14)(23)$	x^2	1423	4		$(14)(23)$
$(134)(2)$	x^2	1342	4		$(134)(2)$
(1234)	x^3	1234			(1234)
$(124)(3)$	x^2	1243	4	1	(1324)
(1324)	$-x^2$	1324	3	2	$(124)(3)$

Table.4. Involution $\psi_{4,4}$

Hence,

$$\begin{aligned} &\sum_{\pi\in\mathfrak{S}_{n,n}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} \\ = &\sum_{\pi\in\mathcal{A}_{n,n}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} + \sum_{\pi\in\mathfrak{S}_{n,n}\setminus(\mathcal{A}_{n,n}\cup\Omega_{n,n})} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} + \sum_{\pi\in\Omega_{n,n}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} \\ = &\sum_{\pi\in\Omega_{n,n}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = x(1+x)^{n-2}. \end{aligned}$$

□

2.2 Cycle Descent Statistics on Derangements

Let $P_n(x, y, 0, 1) = \sum_{i=1}^{n} P_{n,i}(x, y, 0, 1)$. We first give the recurrence for $P_n(x, y, 0, 1)$.

Lemma 2.3. *For any $n \geq 2$ and $2 \leq i \leq n+1$, we have*

$$P_{n+1,i}(x,y,0,1) = xP_{n-1}(x,y,0,1) + x\sum_{j=2}^{i-1} P_{n,j}(x,y,0,1) + y\sum_{j=i}^{n} P_{n,j}(x,y,0,1).$$

Proof. For any $\pi = \pi(1)\pi(2)\ldots\pi(n+1) \in \mathscr{D}_{n+1,i}$, let $\sigma = (1, c_1, c_2, \ldots, c_l)$ be the cycle in the standard cycle decomposition of π which contains the number 1. So π can be split into the cycle σ and a permutation τ on the set $\{1,2,\ldots,n+1\} \setminus \{1, c_1, \ldots, c_l\}$, i.e., $\pi = \sigma \cdot \tau$. Clearly, $l \geq 1$, $i \geq 2$ and $c_l = i$ since $\pi \in \mathfrak{S}_{n+1,i}$. We distinguish between the following two cases:

Case 1. $l = 1$.

Deleting the cycle $(1, c_1) = (1, i)$ from the standard cycle decomposition of π, we obtain the permutation

$$\tau = \pi(2)\ldots\pi(i-1)\pi(i+1)\ldots\pi(n+1)$$

which is defined on the set $\{2, \ldots, i-1, i+1, \ldots, n+1\}$. Note that $\mathrm{red}\,(\tau) \in \mathscr{D}_{n-1}$,

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tau)) + 1 \text{ and } \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tau)).$$

This provides the term

$$xP_{n-1}(x,y,0,1).$$

Case 2. $l \geq 2$.

Suppose that $c_{l-1} = j$ for some $2 \leq j \leq n+1$. Deleting the number $c_l = i$ from the standard cycle decomposition of π, we obtain a permutation

$$\tilde{\pi} = (1, c_1, \ldots, c_{l-1}) \cdot \tau$$

which is defined on the set $\{1, \ldots, i-1, i+1, \ldots, n+1\}$. Note that $\mathrm{red}\,(\tilde{\pi}) \in \mathscr{D}_n$. Moreover, if $c_{l-1} = j \leq i-1$, then

$$\mathrm{red}\,(\tilde{\pi}) \in \mathscr{D}_{n,j}, \mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tilde{\pi})) + 1, \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tilde{\pi})).$$

This provides the term

$$x\sum_{j=2}^{i-1} P_{n,j}(x,y,0,1).$$

If $c_{l-1} = j \geq i+1$, then

$$\mathrm{red}\,(\tilde{\pi}) \in \mathscr{D}_{n,j-1}, \mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tilde{\pi})), \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tilde{\pi})) + 1.$$

This provides the term

$$y \sum_{j=i}^{n} P_{n,j}(x, y, 0, 1).$$

Thus, for any $i \geq 2$ we have

$$P_{n+1,i}(x, y, 0, 1) = xP_{n-1}(x, y, 0, 1) + x \sum_{j=2}^{i-1} P_{n,j}(x, y, 0, 1) + y \sum_{j=i}^{n} P_{n,j}(x, y, 0, 1).$$

□

Theorem 2.4. *For $n \geq 2$, $i \leq n$, we have*

$$P_{n,i}(x, -1, 0, t) = (-1)^{n-i} x^{i-1} t^{i}. \tag{2.3}$$

Proof. Note that

$$\sum_{\pi \in \mathscr{D}_{n,i}} x^{\mathrm{exc}\,(\pi)} (-1)^{\mathrm{cdes}\,(\pi)} t^{\pi^{-1}(1)} = t^{i} \sum_{\pi \in \mathscr{D}_{n,i}} x^{\mathrm{exc}\,(\pi)} (-1)^{\mathrm{cdes}\,(\pi)} = t^{i} P_{n,i}(x, -1, 0, 1).$$

In order to prove the identity (2.3) in Theorem 2.4, it is sufficient to show that

$$P_{n,i}(x, -1, 0, 1) = (-1)^{n-i} x^{i-1} \tag{2.4}$$

for any $n \geq 2$ and $2 \leq i \leq n$.

(First proof of Identity (2.4)) We fist prove Identity (2.4) by induction on the integer n. It is easy to check that

$$P_{2,2}(x, -1, 0, 1) = x.$$

Assume that the formula holds for any $2 \leq k \leq n$. By Lemma 2.3, we have

$$\begin{aligned} P_{n+1,i}(x, -1, 0, 1) &= xP_{n-1}(x, -1, 0, 1) + x \sum_{j=2}^{i-1} P_{n,j}(x, -1, 0, 1) - \sum_{j=i}^{n} P_{n,j}(x, -1, 0, 1) \\ &= x \sum_{j=2}^{n-1} P_{n-1,j}(x, -1, 0, 1) + x \sum_{j=2}^{i-1} P_{n,j}(x, -1, 0, 1) - \sum_{j=i}^{n} P_{n,j}(x, -1, 0, 1) \\ &= \sum_{j=2}^{n-1} (-1)^{n-1-j} x^{j} + \sum_{j=2}^{i-1} (-1)^{n-j} x^{j} - \sum_{j=i}^{n} (-1)^{n-j} x^{j-1} \\ &= (-1)^{n+1-i} x^{i-1} \end{aligned}$$

for any $2 \le i \le n$.

(First proof of Identity (2.4)) Next we give a bijective proof of the explicit formula (2.4) by establishing an involution $\varphi_{n,i}$ on $\mathscr{D}_{n,i}$. Fix $i \in \{2, \ldots, n\}$. By definition, the weight of each $\pi \in \mathscr{D}_{n,i}$ is $(-1)^{\mathrm{cdes}\,(\pi)} x^{\mathrm{exc}\,(\pi)}$, hence the weight of the cyclic permutation

$$\sigma^i = (1, 2, \ldots, i-1, n, n-1, \ldots, i) \in \mathscr{D}_{n,i}$$

is $(-1)^{n-i} x^{i-1}$.

For any $\pi \in \mathscr{D}_{n,i}$, suppose that $\pi = C_1 \ldots C_k$ is the standard cycle decomposition of π and

$$C_k = (c_{k,1}, \ldots, c_{k,s}).$$

We distinguish among the following three cases:

Case 1. $k = 1$ and $C_k = (1, 2, \ldots, i-1, n, n-1, \ldots, i)$.

Then let $\varphi_{n,i}(\pi) = \pi$.

Case 2. $k \ge 2$ and $\mathrm{red}\,(C_k) = (1, 2, \ldots, r-1, s, s-1, \ldots, r)$ for some $r = 2, 3, \ldots, s$.

Suppose that $C_{k-1} = (c_{k-1,1}, c_{k-1,2}, \ldots, c_{k-1,t})$ and $c_{k,j}$ is the largest number in the set $\{c_{k,1}, c_{k,2}, \ldots, c_{k,s}\}$ for some $j \in \{1, 2, \ldots, s\}$. If $c_{k-1,2} < c_{k,j-1}$, then let

$$\varphi_{n,i}(\pi) = C_1 \ldots C_{k-2} \cdot (c_{k-1,1}, c_{k1}, c_{k2}, \ldots, c_{ks}, c_{k-1,2}, \ldots, c_{k-1,t}),$$

and so we have

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\varphi_{n,i}(\pi)) \text{ and } \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\varphi_{n,i}(\pi)) - 1.$$

For example, we consider a permutation $\pi = (1397)(24586) \in \mathscr{D}_{9,7}$. The largest number in the cycle (24586) is 8, and so $j = 4$. Since $c_{1,2} = 3 < c_{2,3} = 5$, we have

$$\varphi_{9,7}(\pi) = (124586397)$$

and

$$exc(\pi) = exc(\varphi_{9,7}(\pi)) = 5, cdes(\pi) = 2, cdes(\varphi_{9,7}(\pi)) = 3.$$

If $c_{k-1,2} > c_{k,j-1}$ then let

$$\varphi_{n,i}(\pi) = C_1 \ldots C_{k-2} \cdot (c_{k-1,1}, c_{k1}, \ldots, c_{k,j-2}, c_{kj} \ldots, c_{ks}, c_{k,j-1}, c_{k-1,2}, \ldots, c_{k-1,t}),$$

and so we have

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\varphi_{n,i}(\pi)) \text{ and } \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\varphi_{n,i}(\pi)) - 1.$$

For example, we consider a permutation $\pi = (1793)(24586) \in \mathscr{D}_{9,3}$. The largest number in the cycle (24586) is 8, and so $j = 4$. Since $c_{1,2} = 7 > c_{2,3} = 5$, we have

$$\varphi_{9,3}(\pi) = (124865793)$$

and

$$exc(\pi) = exc(\varphi_{9,3}(\pi)) = 5, cdes(\pi) = 2, cdes(\varphi_{9,3}(\pi)) = 3.$$

Case 3. $\mathrm{red}\,(C_k) \neq (1, 2, \ldots, r-1, s, s-1, \ldots, r)$ for any $r = 2, 3, \ldots, s$.

There exists a unique index $\tilde{s}$ such that

$$\mathrm{red}\,(c_{k1}, c_{k2}, \ldots, c_{k\tilde{s}}) = 1, 2, \ldots, r-1, \tilde{s}, \tilde{s}-1, \ldots, r$$

for some $r = 2, 3, \ldots, \tilde{s}$ and

$$\mathrm{red}\,(c_{k1}, c_{k2}, \ldots, c_{k,\tilde{s}+1}) \neq 1, 2, \ldots, \tilde{r}-1, \tilde{s}+1, \tilde{s}, \ldots, \tilde{r}$$

for any $\tilde{r} = 2, 3, \ldots, \tilde{s}+1$. It is easy to check $3 \leq \tilde{s} \leq s-1$. Moreover, suppose that c_{kj} is the largest number in the set $\{c_{k1}, c_{k2}, \ldots, c_{k\tilde{s}}\}$. Then we have

$$c_{k,\tilde{s}+1} < c_{k,j-1} \text{ or } c_{k,\tilde{s}+1} > c_{k\tilde{s}}.$$

If $c_{k,\tilde{s}+1} < c_{k,j-1}$ then

$$\varphi_{n,i}(\pi) = C_1 \ldots C_{k-2} \cdot (c_{k1}, c_{k,\tilde{s}+1}, \ldots, c_{ks}) \cdot (c_{k2}, \ldots, c_{k\tilde{s}}),$$

we have

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\varphi_{n,i}(\pi)) \text{ and } \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\varphi_{n,i}(\pi)) + 1.$$

For example, we consider a permutation $\pi = (124586397) \in \mathscr{D}_{9,7}$. Then $\tilde{s} = 6$ and the largest number in the set $\{1, 2, 4, 5, 8, 6\}$ is 8, and so $j = 5$. Since $c_{1,7} = 3 > c_{1,4} = 5$, we have

$$\varphi_{9,7}(\pi) = (1397)(24586)$$

and

$$exc(\pi) = exc(\varphi_{9,7}(\pi)) = 5, cdes(\pi) = 3, cdes(\varphi_{9,3}(\pi)) = 2.$$

If $c_{k,\tilde{s}+1} > c_{k,\tilde{s}}$ then let

$$\varphi_{n,i}(\pi) = C_1 \dots C_{k-2} \cdot (c_{k1}, c_{k,\tilde{s}+1}, \dots, c_{ks}) \cdot (c_{k2}, \dots, c_{k,j-1}, c_{k,\tilde{s}}, c_{kj}, \dots, c_{k,\tilde{s}-1}),$$

we have

$$\text{exc}\,(\pi) = \text{exc}\,(\varphi_{n,i}(\pi)) \text{ and } \text{cdes}\,(\pi) = \text{cdes}\,(\varphi_{n,i}(\pi)) + 1.$$

For example, we consider a permutation $\pi = (124865793) \in \mathscr{D}_{9,3}$. Then $\tilde{s} = 6$ and the largest number in the set $\{1, 2, 4, 8, 6, 5\}$ is 8, and so $j = 4$. Since $c_{1,7} = 7 > c_{1,6} = 5$, we have

$$\varphi_{9,3}(\pi) = (1793)(24586)$$

and

$$exc(\pi) = exc(\varphi_{9,3}(\pi)) = 5, cdes(\pi) = 3, cdes(\varphi_{9,3}(\pi)) = 2.$$

For the case with $n = 4$, we list all π and $\varphi_{n,i}(\pi)$ in Table. 5.

$\pi \in \mathscr{D}_{4,2}$	$\varphi_{4,2}(\pi)$	$\pi \in \mathscr{D}_{4,3}$	$\varphi_{4,3}(\pi)$	$\pi \in \mathscr{D}_{4,4}$	$\varphi_{4,4}(\pi)$
$(12)(34)$	(1342)	$(13)(24)$	(1423)	$(14)(23)$	(1324)
(1342)	$(12)(34)$	(1423)	$(13)(24)$	(1324)	$(14)(23)$
(1432)	(1432)	(1243)	(1243)	(1234)	(1234)

Table. 5. Involutions $\varphi_{n,i}(\pi)$ for $n = 4$

Hence,

$$\sum_{\pi \in \mathscr{D}_{n,i}} x^{\text{exc}\,(\pi)}(-1)^{\text{cdes}\,(\pi)} = (-1)^{n-i}x^{i-1} + \sum_{\pi \in \mathscr{D}_{n,i} \setminus \{\sigma^i\}} x^{\text{exc}\,(\pi)}(-1)^{\text{cdes}\,(\pi)} = (-1)^{n-i}x^{i-1}.$$

□

Chapter 3 Negative Cycle Descent Permutations

In this chapter, we will study negative cycle descent permutations on $\mathfrak{S}_n$ and $\mathscr{D}_n$.

3.1 A Recursion on Negative Cycle Descent Permutations

Recall that

$$b_n(y,1) = \sum_{\pi \in \mathfrak{S}_n} y^{\mathrm{cdes}\,(\pi)}.$$

We derive the recursion satisfied by the sequence $\{b_n(y,1)\}_{n=1}^{\infty}$ as follows.

Theorem 3.1. *For $n \geq 1$, we have*

$$b_{n+1}(y,1) = b_n(y,1) + \sum_{i=1}^{n} b_i(y,1)\binom{n}{i-1}(y-1)^{n-i} \tag{3.1}$$

with the initial condition $b_1(y,1) = 1$

Proof. Suppose that y is a positive integer. Let $\mathfrak{S}_n(y)$ denote the set of pairs $[\pi, \phi]$ such that $\pi \in \mathfrak{S}_n$ and ϕ is a map from the set $CDES(\pi)$ to the set $\{0, 1, \cdots, y-1\}$. It is easy to see that $\mathfrak{S}_n(y)$ is a subset of the wreath product of $\mathbb{Z}_y \wr \mathfrak{S}_n$ and $b_n(y,1) = |\mathfrak{S}_n(y)|$.

For any $[\pi, \phi] \in \mathfrak{S}_{n+1}(y)$, we distinguish between the following two cases:

Case 1. $\pi(1) = 1$.

Let $\tau = \pi(2)\ldots\pi(n+1)$. Then τ is a permutation defined on the set $\{2, 3, \ldots, n+1\}$ and

$$\mathrm{red}\,(\tau) \in \mathfrak{S}_n.$$

Define a map $\phi' : [n] \mapsto \{0, 1, \ldots, y-1\}$ by letting $\phi'(i) = \phi(\mathrm{red}^{\,-1}(i))$ for $i = 1, 2, \ldots, n$. Then

$$[\mathrm{red}\,(\tau), \phi'] \in \mathfrak{S}_n(y),$$

and so this provides the term $b_n(y,1)$.

Case 2. $\pi(1) \neq 1$.

Let $\sigma = (1, c_1, c_2, \ldots, c_l)$ be the cycle in the standard cycle decomposition of π which contains the number 1. So, π is split into the cycle σ and a permutation τ on the set $\{1, 2, \ldots, n+1\} \setminus \{1, c_1, \ldots, c_l\}$, i.e., $\pi = \sigma \cdot \tau$. Clearly, $l \geq 1$ since $\pi(1) \neq 1$.

Note that there is a unique index $k \geq 1$ which satisfies $c_{k-1} < c_k$ and $c_k > c_{k+1} > \cdots > c_l$. For the sequence $c_k \ldots c_l$, if $\phi(c_i) = 0$ for some $k \leq i \leq l-1$ then let k' be the largest index in $\{k, k+1, \ldots, l-1\}$ such that $\phi(c_{k'}) = 0$; otherwise, $k' = k-1$. Let

$$\sigma' = (1, c_1, \ldots, c_{k'}) \text{ and } \pi' = \sigma' \cdot \tau.$$

Then π' is a permutation defined on the set $[n+1] \setminus B$, where

$$B = \{c_{k'+1}, \ldots, c_l\},$$

and

$$\mathrm{red}\,(\pi') \in \mathfrak{S}_{n+1-|B|}.$$

Define a map $\phi' : [n+1-|B|] \mapsto \{0, 1, \ldots, y-1\}$ by letting

$$\phi'(i) = \phi(\mathrm{red}^{-1}(i))$$

for any $1 \leq i \leq n+1-|B|$. Then

$$[\mathrm{red}\,(\pi'), \phi'] \in \mathfrak{S}_{n+1-|B|}(y).$$

Note that $1 \leq |B| \leq n$ and $B \setminus \{c_l\} \subseteq CDES_{n+1}(\pi)$. For any $k \leq i \leq l-1$, let $\theta(c_i) = \phi(c_i)$. Then θ is a map from the set $\{c_{k'+1}, \ldots, c_{l-1}\}$ to $\{1, 2, \ldots, y-1\}$. So there are $\binom{n}{|B|}$ ways to form the set B and $(y-1)^{|B|-1}$ ways to form the map θ. This provides the term

$$\sum_{i=1}^{n} b_{n+1-i}(y,1) \binom{n}{i} (y-1)^{i-1}$$

Hence we derive the recurrence relation

$$\begin{aligned} b_{n+1}(y,1) &= b_n(y,1) + \sum_{i=1}^{n} b_{n+1-i}(y,1) \binom{n}{i} (y-1)^{i-1} \\ &= b_n(y,1) + \sum_{i=1}^{n} b_i(y,1) \binom{n}{i-1} (y-1)^{n-i}. \end{aligned}$$

□

3.2 A Recursion on Negative Cycle Descent Derangements

Recall that

$$b_n(y,0) = \sum_{\pi \in \mathscr{D}_n} y^{\mathrm{cdes}\,(\pi)}.$$

We derive the recursion satisfied by the sequence $\{b_n(y,0)\}_{n=1}^{\infty}$ as follows.

Theorem 3.2. *For $n \geq 1$, we have*

$$b_{n+1}(y,0) = \sum_{i=0}^{n-1} \binom{n}{i} \left[b_{i+1}(y,0) + b_i(y,0)\right](y-1)^{n-i-1} \tag{3.2}$$

with initial conditions $b_0(y,0) = 1, b_1(y,0) = 0$.

Proof. Clearly, we have $b_0(y,0) = 1$ and $b_1(y,0) = 0$. Suppose that y is a positive integer. Let $\mathscr{D}_n(y)$ denote the set of pairs $[\pi, \phi]$ such that $\pi \in \mathscr{D}_n$ and ϕ is a map from the set $CDES(\pi)$ to the set $\{0, 1, \cdots, y-1\}$. Hence $b_n(y,0) = |\mathscr{D}_n(y)|$.

For any $[\pi, \phi] \in \mathscr{D}_{n+1}(y)$, let $\sigma = (1, c_1, c_2, \ldots, c_l)$ be the cycle in the standard cycle decomposition of π which contains the number 1. So, π is split into the cycle σ and a permutation τ on the set $[n+1] \setminus \{1, c_1, \ldots, c_l\}$, i.e., $\pi = \sigma \cdot \tau$. Clearly, $l \geq 1$ since $\pi(1) \neq 1$.

Note that there is a unique index $k \geq 1$ which satisfies $c_{k-1} < c_k$ and $c_k > c_{k+1} > \cdots > c_l$. For the sequence $c_k \ldots c_l$, if $\phi(c_i) = 0$ for some $k \leq i \leq l-1$ then let k' be the largest index in $\{k, k+1, \ldots, l-1\}$ such that $\phi(c_{k'}) = 0$; otherwise, $k' = k-1$.

We distinguish between the following two cases:

Case 1. $k' = 0$

Let

$$B = \{c_1, \ldots, c_l\}.$$

Note that τ is a permutation defined on the set $[n+1] \setminus \{1, c_1, \ldots, c_l\}$ and

$$\mathrm{red}\,(\tau) \in \mathfrak{S}_{n-|B|}.$$

Define a map $\phi' : [n-|B|] \mapsto \{0, 1, \ldots, y-1\}$ by letting

$$\phi'(i) = \phi(red^{-1}(i))$$

for any $1 \leq i \leq n - |B|$. Then

$$[\mathrm{red}\,(\tau), \phi'] \in \mathfrak{S}_{n-|B|}(y)$$

and there are $b_{n-|B|}(y, 0)$ ways to form the pairs $[\mathrm{red}\,(\tau), \phi']$.

Note that $1 \leq |B| \leq n$ and $B \setminus \{c_l\} \subseteq CDES_{n+1}(\pi)$. For any $k \leq i \leq l-1$, let $\theta(c_i) = \phi(c_i)$. Then θ is a map from the set $\{c_k, \ldots, c_{l-1}\}$ to $\{1, 2, \ldots, y-1\}$. So there are $\binom{n}{|B|}$ ways to form the set B and $(y-1)^{|B|-1}$ ways to form the mapping θ.

This provides the term

$$\sum_{i=1}^{n} b_{n-i}(y, 0)\binom{n}{i}(y-1)^{i-1}.$$

Case 2. $k' \geq 1$

Let

$$\sigma' = (1, c_1, \ldots, c_{k'}) \text{ and } \pi' = \sigma' \cdot \tau.$$

Then π' is a permutation defined on the set $[n+1] \setminus B$, where

$$B = \{c_{k'+1}, \ldots, c_l\},$$

and

$$\mathrm{red}\,(\pi') \in \mathfrak{S}_{n+1-|B|}.$$

Define a map $\phi' : [n+1-|B|] \mapsto \{0, 1, \ldots, y-1\}$ by letting

$$\phi'(i) = \phi(\mathrm{red}^{-1}(i))$$

for any $1 \leq i \leq n+1-|B|$. Then

$$[\mathrm{red}\,(\pi'), \phi'] \in \mathscr{D}_{n+1-|B|}(y)$$

and there are $b_{n+1-|B|}(y, 0)$ ways to form the pairs $(\mathrm{red}\,(\pi'), \phi')$.

Note that $1 \leq |B| \leq n-1$ and $B \setminus \{c_l\} \subseteq CDES(\pi)$. For any $k \leq i \leq l-1$, let $\theta(c_i) = \phi(c_i)$. Then θ is a map from the set $\{c_{k'+1}, \ldots, c_{l-1}\}$ to $\{1, 2, \ldots, y-1\}$. So there are $\binom{n}{|B|}$ ways to form the set B and $(y-1)^{|B|-1}$ ways to form the map θ.

This provides the term

$$\sum_{i=1}^{n-1} b_{n+1-i}(y,0)\binom{n}{i}(y-1)^{i-1}.$$

Hence, we have

$$b_{n+1}(y,0) = \sum_{i=1}^{n}\binom{n}{i}b_{n-i}(y,0)(y-1)^{i-1} + \sum_{i=1}^{n-1}\binom{n}{i}b_{n+1-i}(y,0)(y-1)^{i-1}.$$

□

3.3 A Bijection Between Negative Cycle Descent Permutations and Callan Perfect Matchings

In this section, we give a bijection between the negative cycle descent permutations and Callan Perfect Matchings. Moreover, the bijection also maps the negative cycle descent derangements to Callan Perfect Matchings with no vertical lines.

Lemma 3.3. *There is a bijection Θ_n from the set of cyclic negative cycle descent permutation of $[n]$ to the set of connected Callan perfect matchings of $\mathbb{P}_n$.*

Proof. Let (π, ϕ) be a cyclic negative cycle descent permutation of $[n]$. Then there is exactly one cycle C in the standard cycle decomposition of π. Suppose

$$C = (c_1, c_2, \ldots, c_n)$$

where $c_1 = 1$. Erase the parentheses, draw a bar after each element c_i which has sign $+1$, and add a bar before c_1. Regard the numbers between two consecutive bars as “blocks”. So, we decompose (π, ϕ) into a sequence of blocks

$$B_1, B_2, \ldots, B_k.$$

Suppose that the i-th block B_i contains t_i number $b_{i1}, \ldots, b_{it_i}$ with $b_{i1} > \ldots > b_{it_i}$. We construct a perfect matchings M as follows:

- Step 1. For every block B_i, we connect the vertex $(b_{i,j}, 0)$ to the vertex $(b_{i,j+1}, 1)$ as a down-line of M for any $1 \le j \le t_i - 1$.

- Step 2. For any odd integer $i \in \{1, 2, \ldots, k-1\}$, we connect the vertex $(b_{i,t_i}, 0)$ to the vertex $(b_{i+1,t_{i+1}}, 0)$ as an arc of M. For any even integer $i \in \{1, 2, \ldots, k-1\}$, we connect the vertex $(b_{i,1}, 1)$ to the vertex $(b_{i+1,1}, 1)$ as an arc of M.

- Step 3. If k is odd, we connect the vertex $(b_{1,1}, 1) = (1, 1)$ to the vertex $(b_{k,t_k}, 0)$ as a downline of M; otherwise, connect the vertex $(b_{1,1}, 1) = (1, 1)$ to the vertex $(b_{k1}, 1)$ as an arc of M.

It is easy to check that M is connected and has no uplines. So, M is a connected Callan perfect matching. Define Θ_n as a map from the set of cyclic negative cycle descent permutations of $[n]$ to the set of connected Callan perfect matchings of $\mathbb{P}_n$ by letting $\Theta_n(\pi, \phi) = M$. Let (π, ϕ) and (π', ϕ') be two different cyclic negative cycle descent permutations of $[n]$. Then the sequence of blocks of (π, ϕ) and (π', ϕ') are different. This implies $\Theta_n(\pi, \phi) \neq \Theta_n(\pi', \phi')$, and so the map Θ_n is a injection.

Conversely, let M be a connected Callan perfect matching of $\mathbb{P}_n$. Delete the edge incident with the vertex $(1, 1)$ from M, identify two vertices $(i, 0)$ and $(i, 1)$ in M as a new vertex i for each $i = 1, 2, \ldots, n$, denote by $\mathcal{G}^*(M)$ the graph obtained from M. Then the graph $\mathcal{G}^*(M)$ is a path on the vertex set $[n]$ and can be written as

$$a_1 a_2 \ldots a_n$$

where $a_1 = 1$ and the set $\{a_1 a_2, a_3 a_4, \ldots, a_{n-1} a_n\}$ is the edge set of $\mathcal{G}^*(M)$. Draw a bar after each number a_i which satisfies either (1) $i = n$ or (2) there is an arc of M in

$$\{\{(a_i, 0), (a_{i+1}, 0)\}, \{(a_i, 1), (a_{i+1}, 1)\}\},$$

and add a bar before a_1. Regard the numbers between two consecutive bars as "blocks". So, we obtain a sequence of blocks

$$B'_1, B'_2, \ldots, B'_k.$$

We construct a cyclic negative cycle descent permutations (π, ϕ) of $[n]$ as follows:

- Step $1'$. For each block B'_i, we write the numbers in B'_i in decreasing order, denote by τ_i the obtained sequence, and let

$$\pi = (\tau_1, \tau_2, \ldots, \tau_k).$$

- Step $2'$. For any number $j \in [n]$, suppose j is in a block B'_i for some $1 \le i \le k$. If j is the smallest number in B'_i, then let the sign of j be $+1$; otherwise, let the sign of j be -1. In fact, this defines a map ϕ from $[n]$ to $\{+1, -1\}$.

Then (π, ϕ) is a cyclic negative cycle descent permutations of $[n]$. □

Example 3.4. *Let us consider a cyclic negative cycle descent permutation*

$$(1^+6^-4^-3^+2^+8^-7^-5^+)$$

on the set $\{1, 2, \ldots, 8\}$. *We erase the parentheses, draw a bar after each element which has sign* $+1$*, and add a bar before* 1*. Thus we obtain*

$$|1|643|2|875|$$

and the sequence of blocks

$$B_1 = 1, B_2 = 643, B_3 = 2, B_4 = 875.$$

By Stpdf 1,2,and 3 in the proof of Lemma 3.3, we construct the following dot diagram.

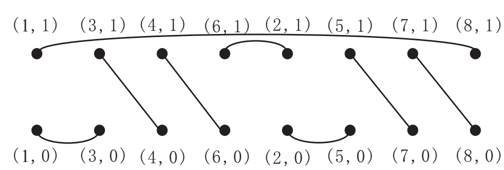


Fig.4. A dot diagram constructed by Step 1,2, and 3 in the proof of Lemma 3.3

Finally, we obtain a connected Callan perfect matching M *corresponding with* $(1^+6^-4^-3^+2^+8^-7^-5^+)$ *as follows:*

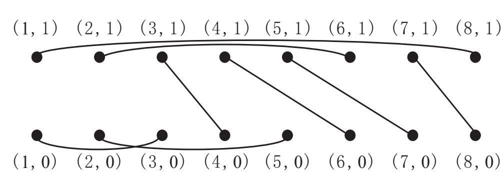


Fig.5. A connected Callan perfect matching M *corresponding with* $(1^+6^-4^-3^+2^+8^-7^-5^+)$

Conversely, let us consider the connected perfect matching M in Fig.5. After deleting the edge $\{(1,1),(8,1)\}$, we can obtain the graph

$$\mathcal{G}^*(M) = 13462578$$

which has the edge set $\{13, 34, 46, 62, 25, 57, 78\}$. Note that there are 3 arcs

$$\{(1,0),(3,0)\}, \{(6,1),(2,1)\}, \{(2,0),(5,0)\}$$

in M. So, we draw bars after the numbers $1, 6, 2, 8$, and add a bar before 1. Thus we obtain

$$|1|346|2|578|$$

and the sequence of blocks

$$B_1' = 1, B_2' = 346, B_3' = 2, B_4' = 578.$$

By Stpdf $1'$ and $2'$ in the proof of Lemma 3.3, we construct a cyclic negative cycle descent permutation

$$(1^+6^-4^-3^+2^+8^-7^-5^+)$$

on the set $\{1, 2, \ldots, 8\}$.

Theorem 3.5. *There is a bijection Γ_n between the set of negative cycle descent permutations of $[n]$ and the set of Callan perfect matchings of $\mathbb{P}_n$. Moreover, for any negative cycle descent permutation (π, ϕ) of $[n]$, we have*

$$\mathrm{com}\,(\Gamma_n(\pi,\phi)) = \mathrm{cyc}\,(\pi),\ \mathrm{ver}\,(\Gamma_n(\pi,\phi)) = \mathrm{fix}\,(\pi),$$

and

$$\mathrm{down}\,(\Gamma_n(\pi,\phi)) = \begin{cases} \mathrm{neg}\,(\pi,\phi) & \textit{if } (1,1) \textit{ and its partner are in the same row,} \\ \mathrm{neg}\,(\pi,\phi)+1 & \textit{otherwise.} \end{cases}$$

Proof. Let (π, ϕ) be a negative cycle descent permutations of $[n]$. Suppose that $\pi = C_1 \ldots C_k$ is the standard cycle decomposition of π and

$$C_i = (c_{i1}, \ldots, c_{i,l_i})$$

for each $i = 1, 2, \ldots, k$. Then

$$\mathrm{red}\,(C_i) \in \mathfrak{S}_{l_i},$$

Define a map $\phi^i : [l_i] \mapsto \{+1, -1\}$ by letting

$$\phi^i(j) = \phi(\mathrm{red}^{\,-1}(j)),$$

i.e., the sign of $\mathrm{red}\,(c_{ij})$ is the same as that of c_{ij}. Then

$$(\mathrm{red}\,(C_i), \phi^i)$$

is a cyclic negative cycle descent permutations of $[l_i]$. By Lemma 3.3, we have $\Theta_{l_i}(\mathrm{red}\,(C_i), \phi^i)$ is a connected Callan perfect matching. For any $1 \leq j \leq l_i$, we replace the labels $(j, 0)$ and $(j, 1)$ of vertices in $\Theta_{l_i}(\mathrm{red}\,(C_i), \phi^i)$ with $(\mathrm{red}^{\,-1}(j), 0)$ and $(\mathrm{red}^{\,-1}(j), 1)$ respectively and denote by M^i the perfect matching obtained from $\Theta_{l_i}(\mathrm{red}\,(C_i), \phi^i)$. At last, let

$$M = M^1 \cup M^2 \cup \ldots \cup M^k$$

where the notation $M \cup M'$ denotes the union of two perfect matchings M and M' such that the vertex set of $M \cup M'$ is $V(M) \cup V(M')$ and the edge set of $M \cup M'$ is $E(M) \cup E(M')$. So M is a Callan perfect matchings of $\mathbb{P}_n$. Define Γ_n as a map from the set of negative cycle descent permutations of $[n]$ to the set of Callan perfect matchings of $\mathbb{P}_n$ by letting $\Gamma_n(\pi, \phi) = M$. Note that Γ_n is injective, and so it is a bijection.

By the definition of Γ_n, it is easy to see that

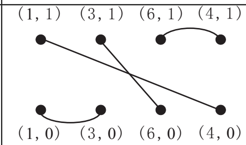
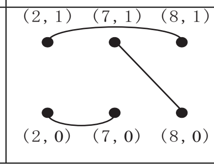
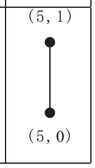
$$\mathrm{com}\,(\Gamma_n(\pi, \phi)) = \mathrm{cyc}\,(\pi) \text{ and } \mathrm{ver}\,(\Gamma_n(\pi, \phi)) = \mathrm{fix}\,(\pi).$$

If the vertices $(1, 1)$ and its partner are in the same row, then $\mathrm{down}\,(\Gamma_n(\pi, \phi)) = \mathrm{neg}\,(\pi, \phi)$; otherwise, $\mathrm{down}\,(\Gamma_n(\pi, \phi)) = \mathrm{neg}\,(\pi, \phi) + 1$. □

Example 3.6. *Let us consider a negative cycle descent permutation*

$$(1^+6^-3^+4^+)(2^+8^-7^+)(5^+)$$

of [8]. *We draw the perfect matchings M^1, M^2 and M^3 corresponding with the cycles C_1, C_2 and C_3 respectively as follows:*

$Cycles$	C_1	C_2	C_3
	$(1^+6^-3^+4^+)$	$(2^+8^-7^+)$	(5^+)
Perfect matchings	M^1	M^2	M^3
			

Finally, we obtain a Callan perfect matching $M = M^1 \cup M^2 \cup M^3$ which is exactly that in Example 1.10.

Let $\Gamma|_{\mathscr{D}_n}$ denote the restriction of Γ_n on the set of negative cycle descent derangements of $[n]$. So the following corollary is immediate.

Corollary 3.7. $\Gamma_n|_{\mathscr{D}_n}$ *is a bijection between the set of negative cycle descent derangements of* $[n]$ *and the set of Callan perfect matchings of* $\mathbb{P}_n$ *which have no vertical lines.*

Proof. $\Gamma_n(\pi)$ has vertical lines if and only if π has cycle with single number thus is not a derangement. □

3.4 A Bijection Between Negative Cycle Descent Derangements and L-S Trees

In this section, we prove that the the number of L-S trees t_n equals to $b_n(2,0)$ for any $n \geq 1$ and give a bijection between L-S trees and negative cycle descent derangements.

Theorem 3.8. *Let $t_0 = 1$ and $t_1 = 0$. Then for $n \geq 1$, we have*

$$t_{n+1} = \sum_{i=0}^{n-1} \binom{n}{i} (t_{i+1} + t_i)$$

and

$$t_n = b_n(2,0)$$

Proof. For any T in $\mathcal{T}_{n+1}$, We first determine $V_1(T)$ of T. If $|V_1(T)| = 1$ then it cannot be $\{n+1\}$ otherwise it contradicts (ii). If $|V_1(T)| = i > 1$, there are $\binom{n+1}{i}$ choices. Whenever $V_1(T)$ is determined, the rest of T forms an element of $\mathcal{T}_{n+1-V_1(T)}$, whose number is given by t_{n+1-i}. Therefore, it leads to the follows:

$$t_{n+1} = nt_n + \sum_{i=2}^{n+1} \binom{n+1}{i} t_{n+1-i} = \sum_{i=0}^{n-1} \binom{n}{i} (t_{i+1} + t_i)$$

□

Lemma 3.9. *There is a bijection Ψ_n between the set of negative cycle descent derangements of $[n]$ and $\mathcal{K}_n$, where $\mathcal{K}_n$ is the set of all trees T rooted at 0, on vertex set $\{0, 1, \ldots, n\}$ satisfying the following two conditions:*

(i) *for $k = 1, 2, \ldots, m(T) - 1$, the only elemmaent of $V_k(T)$ which has children in $V_{k+1}(T)$ is $a_k(T)$;*

(ii) *for $k = 1, 2, \ldots, m(T)$, if $a_k(T) = \min_{i \geq k} a_i(T)$, then $|V_k(T)| > 1$;*

Proof. Let $\mathscr{D}_n(2)$ denote the set of negative cycle descent derangements of $[n]$. Let $\sigma = (1, c_1, c_2, \ldots, c_l)$ be the cycle in the standard cycle decomposition of π which contains the number 1. So, π is split into the cycle σ and a permutation τ on the set $[n+1] \setminus \{1, c_1, \ldots, c_l\}$, i.e., $\pi = \sigma \cdot \tau$. Clearly, $l \geq 1$ since $\pi(1) \neq 1$.

Note that there is a unique index $k \geq 1$ which satisfies $c_{k-1} < c_k$ and $c_k > c_{k+1} > \cdots > c_l$. For the sequence $c_k \ldots c_l$, if $\phi(c_i) = -1$ for some $k \leq i \leq l-1$ then let k' be the largest index in $\{k, k+1, \ldots, l-1\}$ such that $\phi(c_{k'}) = -1$; otherwise, $k' = k - 1$.

We distinguish between the following two cases:

Case 1. $k' = 0$

We define $V_1(T) = \{1, c_1, \ldots, c_l\}$. Then we go back to the start and map the rest part τ to another tree T' according to the two cases. T is constructed as T' is exactly the tree rooted at 1 in T.

Case 2. $k' \geq 1$

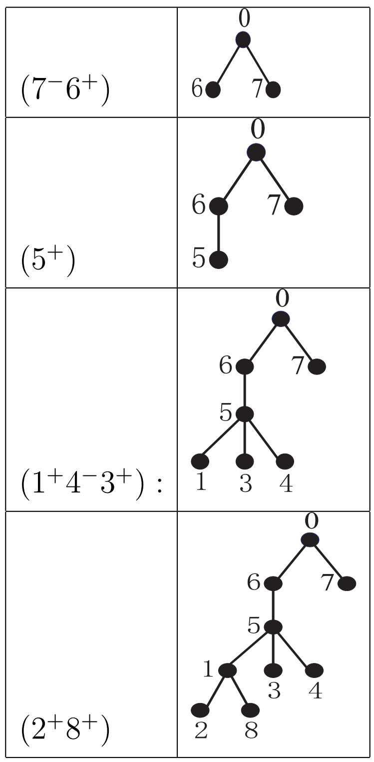
Let

$$\sigma' = (1, c_1, \ldots, c_{k'}) \text{ and } \pi' = \sigma' \cdot \tau.$$

We define $V_1(T) = \{c_k, \ldots, c_l\}$. Then we go back to the start and map the rest part $\pi' = \sigma' \cdot \tau$ to another tree T' according to the two cases. T is constructed as T' is exactly the tree rooted at c_k in T.

Finally, we define $\Psi_n(\pi)$=T as we constructed. Clearly, Ψ_n is an injection thus a bijection. □

Example 3.10. *We pick a negative cycle descent derangement denoted as* $(1^+4^-3^+5^+7^-6^+)(2^+8^+)$



Theorem 3.11. *For $n \geq 2$, there is a bijection Θ_n between the set of negative cycle descent derangements of $[n]$ and $\mathcal{T}_n$.*

Proof. By Lemma 3.9, it is sufficient to prove that $\mathscr{T}_n$ and $\mathscr{K}_n$ are in bijection. We are asked to construct a bijective map $\Delta_n : \mathscr{K}_n \to \mathscr{T}_n$ such that $\Theta_n = \Delta_n \Psi_n$.

Given any T_1 in $\mathscr{K}_n$. we set k to be the smallest number such that $a_k(T) \geq \bigcup_{j \geq k} V_j(T)$ and define a $b_k(T) = \min \bigcup_{j \geq k} V_j(T)$. We list $\bigcup_{j \geq k} V_j(T)$ by order as

$$b_k(T) < i_1 < i_2 < \ldots < i_j < a_k(T)$$

δ_n is defined as the follows.

(i) $\delta_n(a_k(T)) = b_k(T)$;

(ii) $\delta_n(b_k(T)) = i_1,\ \delta_n(i_k) = i_{k+1},\ k = 1, \ldots, j-1,\ \delta_n(i_j) = a_k(T)$;

(iii) $\delta(x) = x, x \notin \bigcup_{j \geq k} V_j(T)$.

Clearly, under several steps limited by n, T_1 will be mapped into $T \in \mathscr{T}_n$. Therefore, we only need to define $\Delta_n = \delta_n^n$. Theorem 3.11 is proved □

Chapter 4 Conclusion

In this thesis, we study the cycle descent statistic of permutations. We obtained several formulas by working on simple examples and numerical tests. Motivated by the these formulas, we shall study the cycle descent statistic of permutations. We define the following enumerative polynomials:

$$P_{n,i}(x,y,q,t) = \sum_{\pi \in \mathfrak{S}_{n,i}} x^{\mathrm{exc}\,(\pi)} y^{\mathrm{cdes}\,(\pi)} q^{\mathrm{fix}\,(\pi)} t^{\pi^{-1}(1)}.$$

In Chapter 2, we calculate the value of the polynomials on permutations and derangements respectively by taking $y = -1$, $q = 1$ and $y = -1$, $q = 0$. we prove our formulas with both induction method and bijection method.

In Chapter 3, based on the generalized Eulerian Polynomials, we define two combinatorial objects on permutations and derangements, namely negative cycle descent permutations and negative cycle descent derangements. In section 3.1 and 3.2, we respectively present a recursion formula for these two objects and construct bijection proofs for the formulas. In section 3.3, we introduce perfect matchings and we construct a bijection between a special type of perfect matchings, Callan perfect matching, and the negative cycle descent permutations. In section 3.4, we introduce the L-S trees, which is proved to be the reduced Euler characteristic for the simplicial complex of bipartite graphs. We first prove the L-S trees and negative cycle descent derangements have the same cardinality and recursion formula. Then, we construct a bijection between the two objects.

In the future, we plan to study the cycle descent statistics on the permutations of multi-set. We believe that more generalized formulas can be obtained on muti-set.

REFERENCE

[1] E. Bagno, A. Butman, and D. Garber, Statistics on the multi-colored permutation groups, Electron. J. Combin. 14 (2007), #R24.

[2] F. Brenti, q-Eulerian polynomials arising from Coxeter groups, *European J. Combin.* 15 (1994), 417–441.

[3] F. Brenti, A class of q-symmetric functions arising from plethysm, *J. Combin. Theory Ser. A,* 91 (2000), 137–170.

[4] D. Callan, Klazar trees and perfect matchings, *European J. Combin.* 31 (2010), 1265–1282.

[5] W.Y.C. Chen, R.L. Tang, and A.F.Y. Zhao, Derangement polynomials and excedances of type B, *Electron. J. Combin.* 16(2) (2009) #R15.

[6] D. Foata, M. Schützenberger, Théorie Géométrique des Polynômes Euleriens, Lecture Notes in Mathematics, vol. 138, Springer-Verlag, Berlin-New York, 1970.

[7] M. Klazar, Twelve countings with rooted plane trees, *European J. Combin.* 18 (1997), 195–210.

[8] G. Ksavrelof, and J. Zeng. Two involutions for signed excedance numbers, *Sém. Lothar. Combin.,* 49 Art. B49e, 2003.

[9] S. Linusson, J. Shareshian, Complexes of t-corolable graphs, *Siam J. Discrete Math.* Vol. 16, No. 3, pp.371–389.

[10] L. Lovász, M. Plummer, Matching theory, Annals of Discrete Mathematics 29, North Holland Publishing Co., Amsterdam, 1986.

[11] Q. Ren, Ordered partitions and drawings of rooted plane trees, *Discrete Math.* 338 (2015), 1–9.

[12] R. P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge Univ. Press, 1999.

[13] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2010.

[14] A.F.Y. Zhao, Excedance numbers for the permutations of type B, *Electron. J. Combin.* 20(2) (2013), #P28.

ACKNOWLEDGEMENT

I would like to say thank you to my Mentor Professor Jun Ma. Most of the results come from our insightful discussion. Also, I would like to say thank you to Professor Yeong-Nan Yeh and Professor Shimei Ma. They also provided very good ideas during our work.

I would like to say thank you to the professors who will look at my thesis for the job they do.

I would like to say thank you to the stuffs who are in charge of the graduation thesis.

I would like to say thank you to every teacher and student from Department of Mathematics and everyone who has had any talk with me or has done any help for my work.