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摘要

对等网络（P2P），基于其公平分配资源的理念，近年来的发展取得了长足的进步。常见的对等网络有供用户分享资源的Bit Torrent以及基于众包的移动终端系统Open Garden。而除了其应用前景外，另一个同样值得思考的事情是，对等网络的参与者能否通过各种操作使自己在遵循某种协议的同时，实现个体利益的更大化。

在本文中，我考虑了一种已经被广泛研究的协议——比例反应协议（PR），讨论了对等网络中的用户在这种协议下可能的通过撒谎获利的情形。特别的，我着重分析了一个用户企图通过谎报他的效用函数的方式来使自己获得更大利益的情况。这种撒谎策略是通过人为地删去自己与特定相邻用户的连接实现的。本文的主要成果即为给出一个非构造性证明，证明在采取比例反应协议的对等网络中任何一个用户都不可能通过这种撒谎策略来提高自己的收益。

本文给出了一种全新的方法，用于研究对等网络的资源分配协议，以及移动终端网络中的众包资源分配协议时，分析网络中用户个体的策略。利用离散数学的方法，本文解决了一个很长时间没有解决的问题——比例反应协议是否具有诚实性。就我所知，这是第一个有实际背景的带宽资源分配网络被证明是具有非平凡的诚实性。

关键词：诚实性，带宽分配，对等网络

CAN BANDWIDTH SHARING BE TRUTHFUL

ABSTRACT

The popular peer to peer (P2P) system has built its success on the protocols that allocate pooled resources fairly, such as in the P2P resource sharing system of Bit Torrent and the crowd-sourcing mobile system of Open Garden. Equally important is the issue whether participants could take the advantage of the protocols by manipulative actions.

I consider a widely studied proportional sharing (PR) protocol and discuss incentives and opportunities of a player to lie for personal gains. Specifically, I focus on analyzing the case that one of player make a strategic move to misreport its utility function by removing one or more of its incident edges. The main result is a proof that a player deviating from the PR protocol by such a move will not make any improvement on its utility eventually.

This result establishes new understanding in the studies of strategic stability of P2P or crowd resource sharing protocols in Internet and mobile networks. Building on discrete mathematical techniques, it resolves a long time unsolved problem whether the proportional response protocol is truthful. It is the first, to the best of my knowledge, non-trivial result on truthful mechanism for a practical network bandwidth resource sharing scheme.

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Key words: Truthful, Bandwidth sharing, Peer to peer system

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Chapter One Introduction

A peer to peer (P2P) networks is a decentralized and distributed network architecture where individual vertices in the network (called "peers") act as both suppliers and consumers of resources. Here the resources are shared among multiple interconnected peers who each uses plum-for-peach (or tit-for-tat) mechanisms to make a portion of their resources (such as processing power, disk storage or network bandwidth) directly available to other network participants without the need of centralized coordination ([18], R. Schollmeier, 2002: P0101-0101). Similarly, the rapid growth of the mobile Internet has also provided an opportunities for wider applications of such P2P systems, such as the one in my consideration, P2P bandwidth sharing among users of the network.

Wu and Zhang ([20], F. Wu and L. Zhang, 2007: P354-363), motivated by P2P application systems such as Bit Torrent, have pioneered a model of proportional sharing with a proof of its fixed point as a market equilibrium of the P2P bandwidth sharing network. A key technique in their study is an elegant mathematical structure on the network called bottleneck decomposition. Iosifidis, et al., in the context of crowd-sourcing mobile net such as Open Garden and Karma ([9], G. Iosifidis, L. Gao, J Huang and L. Tassiulas, 2014: P451-459) have recently proposed incentive schemes for such resource sharing protocol and put forth a challenge of truthful auction mechanism design under this new setting ([10] G. Iosifidis, L. Gao, J. Huang and L. Tassiulas, 2014: P20-27). Despite of the differences in applications, both are tackling a problem under a distributed environment where autonomous agents make their decisions on resource sharing, based on their own rational interests.

My study consider the incentives of network participants, focusing on the proportional response protocol (PR). The dynamics of which was studies in ([20], F. Wu and L. Zhang, 2007: P354-363), where the stable state was shown to converge to a fixed point equivalent to a market equilibrium solution.

More precisely, each vertex represents an agent, who is interested in the bandwidth of its neighbors. The utility of an agent is equal to the total amount of bandwidth it will acquire eventually. At the market each agent sells its own bandwidth. With the money earned through the trade, it is in turn spent to buy bandwidth from neighbors for the agent's own use. The price of the bandwidths may not be the same among all the vertices. It is known that there is always a market equilibrium price vector under this setting, and this Arrow-Debreu market equilibrium is always solvable in polynomial time ([12], K. Jain, 2004: P303-318).

A crucial question comes up then to challenge the protocol in terms of the incentive compatibility of the protocol's participants. It has been long known that incentive compatible protocols and market equilibrium have significant differences in terms of participants' incentives. In particular, for the linear market model, it is known that agents have incentives not always to report their true utilities ([1], B. Adsul, C. S. Babu, J. Garg, R. Mehta and M. A. Sohoni, 2010: P30; [4], N. Chen, X. Deng and J. Zhang, 2011: P106-118). It was shown that some agent may be able to increase its true utility function value by 100% if it reports a false utility function ([5], N. Chen, X. Deng, H. Zhang and J. Zhang, 2012: P464-475). That prompted the definition of the incentive ratio as the maximum multiplicative factor of the utility an agent may gain by lying about its utility function. Though this does not disprove the truthfulness of the PR protocol, it casts a shadow of doubt.

Failing to establish a general result for the incentive compatibility property from the underlying market equilibrium model as pointed out in ([5] , N. Chen, X. Deng, H.

Zhang and J. Zhang, 2012: P464-475), it worth asking whether this can be proven at a more restricted sense. Would the agents motivated to truthful plays when only limited manipulations are possible? In particular, could this be established for the fundamental bandwidth sharing problem?

Then, what would create a possibility of improvement by cheating in the P2P environment under the PR protocol? It is natural for a vertex agent to make a strategic move to misreport its utility function by removing one or more of its incident edges, an event that happens when a network connection is broken, or shut down for different reasons. It may also be faked, for example, as a network service provided by the carriers. In other word, the participating agents may intentionally lie about their communication link if there is an advantage to gain. Seemingly immediately against this potential strategic play, the PR mechanism plays a tit-for-tat counter measure to tackle with that type of cheating. Would the measure be enough to make cheaters non-profitable? I confirm such type of the strategy stability of the market equilibrium under the PR protocol, even though cheating is possible to gain in general for a linear market.

1.1 Technical Contributions

The bottleneck decomposition approach shaped in ([20], F. Wu and L. Zhang, 2007: P354-363) for the convergence of the PR protocol derives a market equilibrium solution immediately. My approach is based on this combinatorial structure. In fact, the relationship between the bandwidth decomposition of the original network and that in the structure altered network plays an important role in establishing the non-increasing property for the utility of the cheating vertex agent who deletes an edge or more.

The network builds on an undirected graph $G = (V, E; w)$ where each vertex owns a bandwidth w_i to be allocated to its neighbors. First a concept of bottleneck is proposed to find the minimum ratio of $\frac{w(\Gamma(B))}{w(B)}$ over all subset $B \subseteq V$, where $\Gamma(B) = \{y : \exists x \in B \; and \; (x, y) \in E\}$ and $w(B) = \sum_{x \in B} w(x)$. It is possible that $B \cap \Gamma(B) \neq \emptyset$. In case of a tie, one of maximal size (equivalently weight) is chosen, and the solution is called a maximum bottleneck. The procedure is applied again after removing B and $\Gamma(B)$, repeatedly until the remaining graph becomes empty.

This elegant structure proves to be very useful in constructing the market equilibrium solution. This structure of bottleneck decomposition can be found in polynomial time proposed in ([20], F. Wu, L. Zhang, 2007: P354-363). For completeness, a proof is placed in the fourth chapter. This direct approach makes a clear intuitive understanding of the final solution.

Those results form the basis for my further discussion on truthful reports by vertex agents on the availability of their communication channels to their neighbors. My comparison of the utilities of a vertex agent in the original network and its altered network is via a concept of the inclusive expansion ratio of set B in the definition of the maximal bottleneck. The study requires an extensive study on the combinatorial properties of the graph with respect to the bottleneck decomposition.

The result is quite unique and different from previous results in the linear market where there are possibilities for an agent to cheat. The P2P setting provides a unique model here. An especially different feature in the cheating behavior I consider here will change the utility function of the cheater as well as its neighbor on the other end of the deleted edge.

1.2 Related Works

General equilibrium has had a deep impact on Economics as a theory. To some extent, its computational aspect has helped to predict the consequences of economic policies such as in the World Bank's computable equilibrium studies ([3], T. Condon, H. Dahl, S. Devarajan, 1986). The allocation of computational resources such as in the cloud computing settings ([19], F. Teng, F. Magoules, 2010: P195-202) has also made equilibria a possible subject of research at the moving frontier of the development of information and communication technology. The computational complexity and approximation issues of market equilibria were initiated in ([6], X. Deng, C. Papadimitriou and S. Safra, 2002: P67-71; [15], C. Papadimitriou, 2001: P749-753). Algorithms of combinatorial nature was especially emphasized by Devanur, et al. ([7], N. R. Devanur and C. H. Papadimitriou, 2002: P389-389) for linear utility functions in Fisher's market. Jain ([12], K. Jain, 2004: P303-318) showed that the Arrow-Debreu market equilibrium with linear utility functions can be computed exactly in a polynomial time, and Y. Ye ([21], Y. Ye, 2008: P315-348) proposed an interior method to solve it.

There is, however, a well-studied issue in the literatures of market equilibria, of agent truthfulness in presenting its utility functions. Under Lange's setting ([13], O. Lange, 1967: P158-61), as an example, of market mechanisms at which the use a huge computer program to compute an equilibrium based on economic agents' submitted utility function, would one obtain a better utility by submitting a fake utility? Even when the price is simulated in a market manner by a sequence of changes in the prices in response to the market demand and supply, an agent may consistently report its demand in accordance to the utility function it wants to pass over to the market

mechanism instead of its true utility function.

The issue has been noticed in Economics community in the past and agent strategic moves on the whole economy have been analyzed with respect to the global economy. Hurwicz ([8], L. Hurwicz, 1972) showed that it is not possible to design an incentive compatible mechanism that guarantees a market equilibrium outcome. After the prices of goods are determined, a buyer choose to buy a bundle that optimize its utilities, and the market equilibrium price vector is one that clears all goods in the market. Therefore, its preference has an effect on the equilibrium price formation. Misrepresenting the preference has an effect in the final equilibrium price and allocation, as well as the cheater's own utility function value. However, such market power becomes negligible in a large market where each participating individual has little influence on the market: a line of research has focused on how incentives for deviating from price-taking behavior decrease as the market grows ([11], M. Jackson and A. Manelli, 1997: P354-376; [14], Y. Otani and J. SSicilian, 1982: P47-68; [16], D. Roberts and A. Postlewaite, 1976: P115-127).

Within the algorithmic game theory community, Adsul, et al. ([1], B. Adsul, C. S. Babu, J. Garg, R. Mehta and M. A. Sohoni, 2010: P30) and Brânzei et al., ([2], S. Brânzei, Y. Chen, X. Deng, A. Filos-Ratsikas, S. K. S. Frederiksen and J. Zhang, 2014: P587-593) explored the issue false report of utility functions by the buyers at a linear Fisher Market and proposed the Nash equilibrium game as a linear utility function game on the market equilibrium outcomes. Chen, et al., ([4], N. Chen, X. Deng and J. Zhang, 2011: P106-118) proposed a concept of incentive ratio. They explored the issue of how much an incentive can be created to lure an individual agent to falsely report its utility functions in the Fisher market. The maximum multiplicative factor, named the incentive ratio, is proven to be exactly two for the linear market. Those negative results on truthfulness under the market equilibrium environment make the design of truthful algorithm impossible, and hence pose a challenge on the

market equilibrium paradigm on how it can take strategic behaviors of buyers and sellers into consideration.

For the crowd resource sharing network communication system, Iosifidis, et al., posed the question whether it is possible to design a truthful mechanism so that the players will truthfully report their sources for the community to share ([10] G. Iosifidis, L. Gao, J. Huang and L. Tassiulas, 2014: P20-27). Under the special setting of P2P network bandwidth sharing, Wu and Zhang ([20], F. Wu and L. Zhang, 2007: P354-363) presented the proportional response protocol as a proxy for modelling the Bit Torrent application and proved that the proportional response protocol converges to an eventual distribution of bandwidth correspondent to a market equilibrium with linear utility functions, i.e. the total received bandwidth from all its neighbors for the participating agents. The result opens up the methodology of market equilibrium as a tool for resource allocation in a distributed management system by selfish and rational agents. At the same time, it also leaves open the possibility of agent manipulation through cheating to deviate from the market equilibrium solution.

The strategic manipulation model of cutting an edge or more is quite natural. It has been well studied in different settings such as kidney exchange RSU, where compatibility of kidneys are desired by the protocol and a key question is whether all the compatible pairs of kidney donor and patients are reported, in order to save as much lives as possible. The setting in my model is slightly different but the game theoretical structure between participants are very closely related.

1.3 Organization of Presentation

This work gives a positive answer to the challenge of truthfulness in the proportional

response protocol for bandwidth sharing in P2P networks. In Chapter Two, I introduce the proportional response protocol, the necessary concept and properties of market equilibrium and bottleneck decomposition of a graph. I also present some computational issues for the bottleneck decomposition and hence one for the market equilibrium for the P2P communication problem. Even though the market equilibrium for the linear market is known in the past, the one derived here is crucial for the eventual stability proof. In particular, the strategic moves by an agent to remove some of its adjacent edges will not only change its own utility but also that of its neighbors. The tight bound of incentive ratio two [5] will then not applicable here. Based on the particular market equilibrium solution derived from the bottleneck decomposition, I prove the stability result in Chapter Three. I develop two main lines of analysis here. One is to utilize the structural properties of the bottleneck decomposition of the underlying graph, as well as that of the altered graph by an agent's cheating actions. Another is the incentive analysis of the cheating agent. Those two approaches develop a set of combinatorial structures delimiting the possible structures to a small set of possible networks. After that, I give another approach to proof the main theorem in Chapter Four, as an implement for just showing another train of thought when dealing with structure of a graph. Finally, in Chapter Five, I talk about one of possible future works based on this paper, the incentive ratio of bandwidth sharing, and also conclusion in Chapter Six.

Chapter Two Definition and Terminologies

In this section, I introduce basic concepts of a P2P network model in the context of bandwidth sharing, as established in ([20], F. Wu and L. Zhang, 2007: P354-363), to which most of the related materials in this section is referred.

First, a protocol of fair allocation is introduced, and in comparison, an economic solution of market equilibrium. Next, I discuss the properties of a bottleneck decomposition structure of a network associated with the market equilibrium solution. Finally, I derive the market equilibrium solution from the bottleneck decomposition and their computation.

2.1 P2P Bandwidth Sharing

Consider a connected network, represented by an undirected graph $G = (V, E)$. Each vertex $v \in V$ represents a player with an upload bandwidth capacity $w(v) > 0$ to be shared with its neighbors. Each edge represents a communication link. Let $\Gamma(v)$ denote the set of vertices adjacent to v in graph G and x_{uv} denote the fraction of bandwidth v allocated for use by its neighbor u. Therefore, the upload bandwidth a vertex v provides to a neighboring vertex u is $w_v \times x_{uv}$. $\mathrm{X} = (x_{uv})_{u,v \in V}$ is called an allocation. And the vertex agents are also used to refer to the players.

As follows, the proportional response protocol is based on a particular fairness condition.

2.1.1 Proportional Response Protocol

For each vertex u, the allocation $(x_{uv} : v \in \Gamma(u))$ of its bandwidth w_u is proportional to what it receives from its neighbors $(w_v \times x_{vu} : v \in \Gamma(u))$. The protocol is known to have a fixed point.

Alternatively, one may consider an economy model where each player i sells its own bandwidth to its neighbors, and at the same time each player is also a buyer of bandwidth from its neighbors for its own communication needs. It is assumed that each player's utility is the sum of bandwidth it acquires from its neighbors, i.e., the utility of *v* for an allocation *X* is $U_v(X) = \sum_{u\in\Gamma(v)} x_{uv}w_u$. One supplies its bandwidth only for the communication needs of its neighbors. We now define the market equilibrium in the economy model.

2.1.2 Market Equilibrium

Let p_i be the price player i of its whole bandwidth, $i = 1,2,\cdots,n$. The price vector $p = (p_1,\cdots,p_n)$, together with the allocation $X = (x_{uv})$ is called a market equilibrium if $\forall v:,1 \le v \le n$ the following holds:

(1) Market clearance: $(\sum_{u\in\Gamma(v)} x_{uv} - 1)\cdot p_v = 0$. All the bandwidth of *v* are allocated or priced null.

(2) Budget constraint: $\sum_{u\in\Gamma(v)} x_{uv}p_u \le p_v$. The money player v should pay out must not exceed its budget, i.e., what it gets from selling its own bandwidth at the market.

(3) Individual optimality: The solution $X = (x_{uv})$ maximizes utility $\sum_{u\in\Gamma(v)} x_{uv}w_u$, subject to $\sum_{u\in\Gamma(v)} x_{uv}p_u \le p_v$ and $x_{uv} \ge 0$ for each vertex *v*. That is, each player is optimally happy for its allocation at the current price.

For the market clearance condition, the bandwidth one owns is either fully sold to its neighbors or its price is set to zero. The latter may occur such as in the case of an independent vertex in the P2P bandwidth sharing problem.

2.2 Bottleneck Decomposition

Consider an undirected graph $G = (V, E; w)$ where $w : V \to R_+$ is a bandwidth function on each vertex *v*. For any set $S \subseteq V$, we define $w(S) = \sum_{u \in S} w(u)$ and $\Gamma(S) = \bigcup_{v \in S} \Gamma(v)$ where $\Gamma(v)$ is the neighborhood of v in G. It is possible that $S \cap \Gamma(S) \neq \emptyset$. Define $\alpha(S) = \frac{w(\Gamma(S))}{w(S)}$ referred to as the inclusive expansion ratio of *S*, or the α-ratio of *S* for short.

2.2.1 Maximal Bottleneck

A vertex subset $B \subseteq V$ is called a *bottleneck* of G if $\alpha(B) = \min_{S \subseteq V} \alpha(S)$. *B* is a maximal bottleneck if for any subset $\widetilde{B}$, we have $B \subsetneq \widetilde{B} \subseteq V$ implies $\alpha(\widetilde{B}) > \alpha(B)$. We also call $(B, \Gamma(B))$ the maximal bottleneck pair.

2.2.3 Bottleneck Decomposition

Given $G = (V, E; w)$. Start with $V_1 = V, G_1 = G$ and $i = 1$. Find the maximal bottleneck B_i of G_i and let $G_{i+!}$ be the induced subgraph on the vertex set$V_{i+1} = V_i - (B_i \cup C_i)$, where $C_i = \Gamma(B_i) \cap V_i$, the neighbour set of *Bi* in the subgraph G_i. Repeat if $G_{i+1} \neq \emptyset$ and set $k = i$ if $G_{i+1} = \emptyset$. Then we call $\mathcal{B} = \{(B_1, C_1), \cdots, (B_k, C_k)\}$ the *bottleneck decomposition* of *G*, α_i the i-th α-ratio and $< \alpha_i = \frac{w(C_i)}{w(B_i)} : i = 1; 2; \cdots; k >$ the α-ratio vector.

With these definition, we can get basic properties of the *bottleneck decomposition*, some of which will be useful in my work and shown as following.

Proposition 1. ([20], F. Wu and L. Zhang, 2007: P354-363) Given graph G, we have:

1. $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_k \le 1$;

2. If there is a pair (B, C) in G_i such that $\frac{w(C)}{w(B)} = \alpha_i$, then $B \subseteq B_i$ and $C \subseteq C_i$;

3. If $\alpha_i = 1$, then $i = k$ and $B_i = C_i$; otherwise B_i is an independent set and $B_i \cap C_i = \emptyset$;

4. $\bigcup_{i:\alpha_i<1} B_i$ is an independent set of vertices;

5. The *bottleneck decomposition* is unique.

Proposition 2. ([20], F. Wu and L. Zhang, 2007: P354-363) For the bottleneck decomposition $\mathcal{B} = \{(B_1, C_1); \cdots ; (B_k, C_k)\}$ of G:

1. There is no edge between B_i and B_j , $i \ne j = 1,2, \cdots k$;

2. There is no edge between B_i and C_j, where $j > i$.

With those properties, a connection was established in ([20], F. Wu and L. Zhang, 2007: P354-363) from the bottleneck decomposition to the market equilibrium. First a price vector is established:

1. for any $u \in B_i$, let $p_u = \alpha_i w_u$;

2. for any $u \in C_i$, let $p_u = w_u$;

3. If $\alpha_k = 1$, then for any $u \in B_k = C_k$, let $p_u = w_u$.

The corresponding allocations of bandwidth of the vertices are determined by the bottleneck decomposition, divided into three cases:

(1) For $\alpha_i < 1$, consider the bipartite graph $\hat{G}_i = (B_i, C_i; E_i)$ where $E_i = (B_i \times C_i) \cap E$. By Proposition 1-(2), B_i is an independent set in G. Furthermore the definition of maximal bottleneck implies that for any subset $B' \subseteq B_i, \alpha(B') > \alpha_i$. Let $\hat{x}_{uv}$ be the amount of bandwidth that vertex $u \in B_i$ upload to $v \in C_i$ along edge $(u, v) \in E_i$. By the max-flow min-cut theorem, there exist $\hat{x}_{uv} \ge 0$ for $u \in B_i$ and $v \in C_i$ such

that $\sum_{v\in\Gamma(u)\cap C_i} \hat{x}_{uv} = w_u$ and $\sum_{u\in\Gamma(v)\cap B_i} \hat{x}_{vu} = w_v/\alpha_i$. Let $\hat{x}_{vu} = \alpha_i\, \hat{x}_{vu}$ which means that $\sum_{u\in\Gamma(v)\cap B_i} \hat{x}_{vu} = w_v$.

(2) When $B_k = C_k$ with $\alpha_k = 1$, similarly we construct a bipartite graph $\hat{G}_k = (B_k, B_k'; E_k')$ such that B_k' is a copy of B_k, there is an edge $(u, v') \in E_k'$ iff $u, v \in B_k$ and $(u, v) \in E[B_k]$. Then, by Hall's theorem, for any edge $(u, v) \in E[B_k]$, there exist $\hat{x}_{uv'}$ such that $\sum_{v'\in\Gamma(u)\cap B_k'} \hat{x}_{uv'} = w_u$.

(3) For any other edge, $(u, v) \notin (B_i \times C_i) \cup (C_i \times B_i), i = 1,2,\cdots,k$, define $\hat{x}_{uv} = 0$.

It is clear that the allocation defined above assigns all the bandwidth of each vertex to its neighbors, we call it a feasible allocation. For this particular feasible allocation, all available bandwidth are uploaded along edges in $(B_i \times C_i) \cup (C_i \times B_i), i = 1,2,\cdots,k$. In addition, if we define $x_{uv} = \hat{x}_{uv}/w_u$ and $X = (x_{uv})$, then

Proposition 3. ([20], F. Wu and L. Zhang, 2007: P354-363) (p, X) is a market equilibrium. Specially, for any vertex $u \in B_i$, and $v \in C_i, i = 1,2,\cdots,k$, the utilities of u and v are $U_u = w(u) \cdot \alpha_i$ and $U_v = w(u)/\alpha_i$, respectively.

The price of the bandwidth is cheaper if a vertex is in the class B_i, than any vertex in $B_j, j > i$, or in $C_t, 1 \le t \le k$. But a vertex in Bi is not accessible from any vertex in $B_j \cup C_j, j > i$. The above pricing and allocation is therefore a market equilibrium.

Computing a market equilibrium can now be reduced to one of the bottleneck decomposition.

2.3 Computation of Bottleneck Decomposition

Since the linear utility Arrow-Debreu equilibrium is solvable in polynomial time ([12], K. Jain, 2004: P303-318), it immediately derives a polynomial time solution for the bottleneck decomposition problem. However, the structure of bottleneck decomposition allows for a much intuitive mathematical interpretation for the analysis of agent incentives. It was known ([20], F. Wu and L. Zhang, 2007: P354-363) that it can be obtained by a parameterized network flow algorithm to compute the bottleneck decomposition of G, directly without going through the linear utility equilibrium. It finds a maximal bottleneck in current graph first, and then we can recursively the algorithm iteratively to the rest of graph. We give a version of this approach originated in ([20], F. Wu and L. Zhang, 2007: P354-363) to prepare for our discussion for the truthfulness of this protocol in the next section.

2.3.1 Derive the Bottleneck Decomposition

We find the bottleneck decomposition by iteratively computing $\alpha_i, B_i, C_i, i = 1,2,\cdots,k$.

Each step is one of finding a maximal bottleneck. Each step is divided into two phases. At the first phase, we shall compute the minimal α-ratio of the current graph. At the second phase, the maximal subset with the same minimal α-ratio shall be constructed.

2.3.2 Evaluating the Minimal α-ratio

Following the ideas in ([20], F. Wu, L. Zhang, 2007: P354-363), we guess a range for

the α-ratio. Then we split the range in halves at each iteration. The max flow algorithm is applied to decide which half of the range will contain the minimum threshold.

Consider a graph $G = (V, E)$ and a parameter α. The new network $N(G, \alpha)$ with $V_N = \{s, t\} \cup V \cup \tilde{V}$ has $\tilde{V}$ as a copy of V. The edge set E_N of N is constructed as follows:

(1) For any vertex $v \in V$, create a directed edge $(s, v) \in \tilde{E}$ from source *s* to *v* with capacity αw_v;

(2) For any vertex $\tilde{v} \in \tilde{V}$, create a directed edge $(\tilde{v}, t) \in \tilde{E}$ from $\tilde{v}$ to sink *t* with capacityw_v;

(3) For any edge $(u, v) \in E$, create two directed edges $(u, \tilde{v}), (v;, \tilde{u}) \in E$ with capacity $+\infty$.

The following results are established to make the binary search approach work. Here $\tilde{B} = \{\tilde{v} | v \in B\}$ and $\Gamma(\tilde{B}) = \{\tilde{v} : (\tilde{u}, \tilde{v}) \in \tilde{E})$ where the latter is called the neighborhood of $\tilde{B}$.

Proposition 4. For any $s - t$ cut (S, T) in network $N(G, \alpha)$, the capacity of cut (S, T) is finite if and only if $S = \{s\} \cup B \cup \Gamma(\tilde{B})$ for any subset $B \subseteq V$. This capacity is $\alpha(w(V) - w(B)) + w(\Gamma(B))$.

Proof. First, if $S = \{s\} \cup B \cup \Gamma(\tilde{B})$, there are two kinds of edges in cut (S, T), we must have:

(1) Edge (s, u) for any $u \in B$ and the total capacity of these edges is $\alpha(w(V) - w(B))$;

(2) Edge $(\tilde{v}, t)$ for any $\tilde{v} \in \Gamma(\tilde{B})$ and the total capacity of these edges is $w(\Gamma(B))$.

So the capacity of cut (S, T) is $\alpha(w(V) - w(B)) + w(\Gamma(B))$, which is finite.

Conversely, if $S \neq \{s\} \cup B \cup \Gamma(\tilde{B})$, then cut (S, T) must contain at least one of two types edge: $(u, \tilde{v})$ or $(v, \tilde{u})$ for an edge $(u, v) \in E$. Thus the capacity of (S, T) is infinite. It's a contradiction.■

Let α^* be the minimal α-ratio of all subsets of vertices in G. By Proposition 4, we know that any cut (S, T) with finite capacity corresponds to a subset $B \subseteq V$. Next lemma will tell us how to improve α to α^*.

Proposition 5. Given network $N(G, \alpha)$, let α^* be the minimal α-ratio of all subsets of vertices in graph G and cap^* be the minimal cut capacity of network $N(G, \alpha)$, then we have:

(1) $cap^* < \alpha w(V)$, if and only if $\alpha > \alpha^*$;

(2) $cap^* > \alpha w(V)$, if and only if $\alpha < \alpha^*$;

(3) $cap^* = \alpha w(V)$, if and only if $\alpha = \alpha^*$.

Proof. Since the proofs for those results are similar, we show the correctness of the first one in details. Let (S^*, T^*) be the min cut of the current network $N(G, \alpha)$ and B^* be the subset in G satisfying that $\frac{w(\Gamma(B^*))}{w(B^*)} = \alpha^*$. By Proposition 4, we know that $S^* = \{s\} \cup B \cup \Gamma(\tilde{B})$ for some set $B \subseteq V$ and its capacity is $cap^* = \alpha\big(w(V) - w(B)\big) + w(\Gamma(B))$. The condition $cap^* < \alpha w(V)$ implies that $\frac{w(\Gamma(B))}{w(B)} < \alpha$. In addition, $\alpha^* < \alpha$ because $\alpha^* < \frac{w(\Gamma(B))}{w(B)}$.

Conversely, if $\alpha^* < \alpha$, we construct a cut (S, T) where $S = \{s\} \cup B \cup \Gamma(\tilde{B})$ based on the subset $B^* \subseteq V$ with $\frac{w(\Gamma(B^*))}{w(B^*)} = \alpha^*$. Then, the capacity of (S, T) is $cap(S, T) =$

$$\alpha(w(V) - w(B^*)) + w(\Gamma(B^*)) = \alpha w(V) + (\alpha^* - \alpha)w(B^*) < \alpha w(V).$$

Therefore, $cap^* < cap(S,T) < \alpha w(V)$.■

It is assumed that the bandwidth amount (or the vertex weight) w_v each player owns to be integers. Otherwise, we can always scale them appropriately to make them integers.

Algorithm A: Computing the minimal α-ratio.

Input: Graph G;

Output: The minimal α-ratio $\alpha^*$of G.

Step 1. Set $a := 0, b := 1$ and $M := w^2(V)$;

Step 2. Set α := $\frac{1}{2}(a+b)$;

Step 3. Build network $N(G,\alpha)$;

Step 4. Compute the min cut (S^*, T^*) with its capacity cap^* and obtain the corresponding subset $B \subseteq V$;

Step 5. If $|\frac{w(\Gamma(B))}{w(B)} - \alpha| < \frac{1}{M}$, then stop and output $\alpha^* = \frac{w(\Gamma(B))}{w(B)}$;

Step 6. Otherwise,

Step 7. If $cap^* > \alpha w(V)$, then set $a := \alpha, b := b$ and turn to Step 2;

Step 8. If $cap^* < \alpha w(V)$, then set $a := a, b := \alpha$ and turn to Step 2.

Theorem 6. ([20], F. Wu and L. Zhang, 2007: P354-363) The minimum α-ratio can be computed in polynomial time for a peer-to-peer network with all integer capacities.

2.3.3 Find the Maximal Bottleneck

Although a subset $B \subseteq V$ with minimal α-ratio α^* can be found in the above, it is possible that such a set B may not be maximal. Since we have $cap(S,T) =$

$\alpha w(V) + (\alpha^* - \alpha)w(B^*)$. Choose $\alpha = \alpha^* - \epsilon$, we obtain $cap(S,T) = \alpha w(V) - \epsilon w(B^*)$, which is minimized on the maximal bottleneck with appropriately chosen ϵ, such as $\frac{1}{M}$.

Chapter Three Incentive Compatibility in the Bandwidth Sharing

In Chapter two, a relationship between the market equilibrium and the bottleneck decomposition is established. Such a bandwidth allocation approach then derives a market equilibrium as a resource allocation rule for P2P network bandwidth resources. A question occurs that, as a distributed network protocol, a player may or may not follow the protocol at the execution level. Can players make strategic moves by misrepresenting their utility functions for gains in their true utilities?

Specific to the bandwidth allocation problem, we consider such a way to lie that one player may do by removing edges adjacent to itself. For simplicity of the presentation, we consider the case one edge is removed. We comment that this would not reduce the generality of the result as we can remove edges one by one at none of which steps the true utility of the cheater would be proven to increase.

For simplicity of the presentation, we consider the case one edge is removed. We observe that this would not reduce the generality of the result as we can remove edges one by one at each of which steps the true utility of the cheater would be proven to be non-increasing.

To prepare for the discussion, we first present two simple cases who are easy to analyze in order to make the problem itself seem more understandable for readers.

Then we give several useful structure properties important to our analysis in Subsection 3.1, including the Basic lemma, the Key Lemma and the Main Lemma. Next, we will study the incentive properties of an agent who would like to cut an incident edge to increase its utility in the bottleneck decompositions of G and $G' = G - (u, v)$ in the original and the resulting network before or after an agent may change the network by deleting an edge of its own.

3.1 Two simple example for the problem

Here are two example to make the problem itself easy to understand. First let us consider a graph of complete graph $G = K_3$. Let us denote $u = 1,2,3$, and let $w_u = 10^u$, then according to the definition of bottleneck decomposition, we can decompose G to get $\mathcal{B} = \{(B_1, C_1)\}$ and $B_1 = \{3\}, C_1 = \{1,2\}$ as well as $\alpha_1 = \frac{11}{100}$. According to Proposition 2, we know that:

(1)Agent 1 gives all its 10 bandwidth to Agent 3;

(2)Agent 2 gives all its 100 bandwidth to Agent 3;

(3)Agent 3 gives its $1000 \times \frac{10}{10+100} = \frac{1000}{11}$ bandwidth to Agent 1;

(4)Agent 3 gives its $1000 \times \frac{100}{10+100} = \frac{10000}{11}$ bandwidth to Agent 2.

Meanwhile, according to Proposition 3, or derived directly from the previous result, we can get that:

(1)Agent 1 gets a utility of $\frac{10}{\frac{11}{100}} = \frac{1000}{11}$;

(2)Agent 2 gets a utility of $\frac{100}{\frac{11}{100}} = \frac{10000}{11}$;

(3)Agent 3 gets a utility of $1000 \times \frac{11}{100} = 110$.

Now consider for instance agent 1 (or agent 2) break its connection with agent 2 (or agent 1, respectively) in purpose. Such a strategic play is equivalent to the case that we delete the edge (1,2) to get a new graph $G' = G - (1,2)$. If this happens, then the new bottleneck decomposition $\mathcal{B}'$ is exactly the same as $\mathcal{B}$ of G, which means for agent 1 (or 2) cannot get more utility from this behavior.

Things are a little different if another edge is deleted but the conclusion is the same. For instance if agent 1 (or agent 3) break its connection with agent 3 (or agent 1, respectively) in purpose, that is $G' = G - (1,3)$, then it is easy to check that new bottleneck decomposition $\mathcal{B}' = \{(\{1,3\},\{2\})\}$ and the α-value of each vertex is $\frac{10}{101}$. As a result, agent 1 get a utility of $10 \times \frac{10}{101} = \frac{100}{101} < \frac{1000}{11}$, and agent 3 get a utility of $1000 \times \frac{10}{101} = \frac{10000}{101} < 110$. So agent 1 (or agent 3, respectively) cannot get more utility from such a strategic play.

Similarly, if agent 2 (or agent 3) break its connection with agent 3 (or agent 2, respectively) in purpose, that is $G' = G - (2,3)$, then it is easy to check that new bottleneck decomposition $\mathcal{B}' = \{\{2,3\},\{1\}\}$ and the α-value of each vertex is $\frac{1}{110}$. As a result, agent 2 get a utility of $100 \times \frac{1}{110} = \frac{10}{11} < \frac{10000}{11}$, and agent 3 get a utility of $1000 \times \frac{1}{110} = \frac{1000}{11} < 110$. So agent 2 (or agent 3, respectively) cannot get more utility from such a strategic play.

Now we have considered all the three cases of deleting an edge of $G = K_3$, and none of them will lead to an increase of utility for the vertex of the edge.

Another example is the graph showing in the Figure 1.

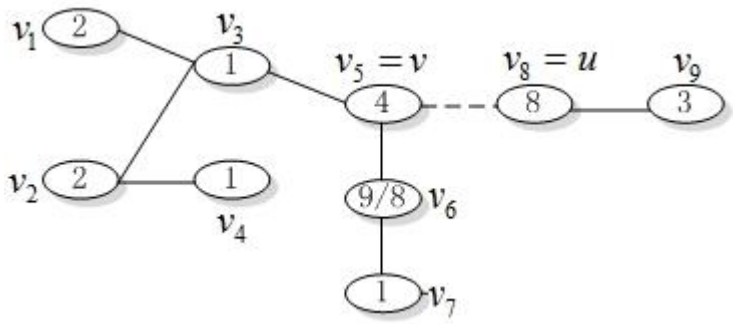

Figure 1: A rather complicated graph for explaining the problem we consider.

In the original graph G there is an edge between $v_5 = v$ and $v_8 = u$. According to the definition of the bottleneck decomposition, we can get $\mathcal{B} = \{(B_1, C_1), (B_2, C_2), (B_3, C_3)\}$. In precise, $B_1 = \{v_1, v_2\}, C_1 = \{v_3, v_4\}, B_2 = \{v_8\}, C_2 = \{v_5, v_9\}, B_3 = \{v_6\}$ and $C_3 = \{v_7\}$. And correspondingly $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{7}{8}$ and $\alpha_3 = \frac{8}{9}$. So it is easy to check that $\alpha_{v_8} = \alpha_{v_5} = \alpha_2 = \frac{7}{8}$.Then v_5 get a utility of $\frac{w_5}{\frac{7}{8}} = \frac{4}{\frac{7}{8}} = \frac{32}{7}$ and v_8 get a utility of $w_8 \times \frac{7}{8} = 8 \times \frac{7}{8} = 7$.

After deleting edge (v_5, v_8) to get a new graph G', its bottleneck decomposition $\mathcal{B}' = \{(B_1', C_1'), (B_2', C_2'), (B_3', C_3')\}$. In precise, $B_1' = \{v_1, v_5, v_7\}, C_1' = \{v_3, v_6\}, B_2' = \{v_8\}, C_2' = \{v_9\}, B_3' = \{v_2\}$ and $C_3' = \{v_4\}$. And correspondingly $\alpha_1' = \frac{1+\frac{8}{9}}{7} = \frac{17}{56}, \alpha_2' = \frac{3}{8}$ and $\alpha_3' = \frac{1}{2}$. So it is easy to check that $\alpha_{v_5}' = \alpha_1' = \frac{17}{14}$ and $\alpha_{v_8}' = \alpha_2' = \frac{3}{8}$.Then v_5 get a new utility of $w_5 \times \frac{17}{56} = 4 \times \frac{17}{56} = \frac{17}{14}$ and v_8 get a new utility of $w_8 \times \frac{3}{8} = 8 \times \frac{3}{8} = 3$. Both v_5 and v_8 get less utility after deleting the edge (v_5, v_8). That is to say the strategic play of agent v_5 (or agent v_8) that it misreports a connection to agent v_8 (or agent v_5, respectively) cannot result in an improvement for its own utility.

3.2 Structure Properties of Bottleneck Decomposition $\mathcal{B}$ and $\mathcal{B}'$

Definition 1 (Bottleneck Decomposition). Let $\mathcal{B} = \{(B_1, C_1), \cdots, (B_k, C_k)\}$ be the bottleneck decomposition of the original graph G and let the α-ratio of (B_i, C_i) be $\alpha_i = \frac{w(C_i)}{w(B_i)}, i = 1,2, \cdots, k$. For pair (B_i, C_i) with $\alpha_i < 1$, each vertex in B_i is called a B-class vertex, and each vertex in C_i is called a C-class vertex. For the special case $B_k = C_k$, i.e., $\alpha_k = 1$, all vertices in B_k are categorized as both B-class and C-class. Define $V_1 = V$, $V_{i+1} = V_i - (B_i \cup C_i)$ for $i = 1, 2, \cdots, k - 1$ and G_i for the induced subgraph on $V_i, i = 1, 2, \cdots, k$. Similarly, for the new graph G' after removing one edge (u, v) from G, its bottleneck decomposition may be different from that of G. Let it be $\mathcal{B}' = \{(B'_1, C'_1), \cdots, (B'_k, C'_k)\}$ with α-ratio $\alpha'_i, i = 1, 2, \cdots, k'$. Similarly, $V'_1 = V, V'_{i+1} = V'_i - (B'_i \cup C'_i)$ for $i = 1, 2, \cdots, k' - 1$ and $G'_i = G'[V'_i], i = 1, 2, \cdots, k'$. Call vertices in $B'_i, 1 \le i \le k'$, B'-class, and those of C'_i's, C'-class.

Therefore, a vertex in V_k with $\alpha_k = 1$ (or $\alpha'_{k'} = 1$) could simultaneously be B-class and C-class (or B'-class and C'-class), in the case $B_k = C_k$ (or $B'_{k'} = C'_{k'}$).

By Proposition 1, B-class vertices, except those in B_k with $\alpha_k = 1$, form an independent subset of vertices. Edges are either between vertices from B_i to C_j $(i \ge j)$, or between vertices from C_i to C_j. Thus at least one of u and v must be a C-class vertex in G. The following is an immediate result of the definition of the maximal bottleneck.

Lemma 1. Given pair (B_i, C_i), if $B \subseteq V_i$ and $C = \Gamma(B) \cap V_i$ then $\frac{w(C)}{w(B)} \ge \alpha_i$. Specially, if $B \subseteq B_i$ and $C = \Gamma(B) \cap C_i$, then $\frac{w(C)}{w(B)} \ge \alpha_i$. Further, if let $B^c = B_i - B$ and $C^c = C_i - C$, then $\frac{w(C^c)}{w(B^c)} \le \alpha_i$.

Definition 2 (Edge (u, v) and Indices $i_u,\ i_v, j_u,\ j_v$). Let $(u, v) \in E$ be cut to obtain $G' = G - (u, v)$. If u and v are in different classes, w.l.o.g, we assume v is in C-class. Let vertex u (and v, respectively) appear in pair (B_l, C_l)at step $l = i_u$ (and $l = i_v$, respectively) of the bottleneck decomposition of G. Specifically, if both u and v are both in C-class, let $i_v \leq i_u$. Similarly, let vertex u (and v, respectively) appear in pair (B'_l, C'_l) at step $l = j_u$ (and $l = j_v$, respectively) of the bottleneck decomposition of G'. Define $j_* = min\{j_u, j_v\}$. In addition, let $\Gamma'(x)$ be the neighborhood of x in G' and define $\Gamma'(S) = \bigcup_{x \in S} \Gamma'(x)$ for set S.

Based on Definition 1 and 2, the following results can be derived directly.

Lemma 2 (Basic Lemma). For the bottleneck decompositions $\mathcal{B}$ and $\mathcal{B}'$,

1. for any $1 \leq t < j_*, (B'_t, C'_t) = (B_t, C_t)$.
2. for any $1 \leq t \leq j_*,\ V'_t = V_t$.
3. $\alpha'_t \leq \alpha_t$.
4. for any $1 \leq t < j_*, B'_t \cap \left(\bigcup_{i=1}^{k} C_i\right)$.
5. for any $1 \leq t < j_*, B_t \cap \left(\bigcup_{i=1}^{k} C'_i\right)$.
6. $j_* \leq i_v \leq i_u$.

Proof. We first show the correctness of Item 1. Suppose to the contrary that $j < j_*$ is the smallest index such that $\left(B'_j, C'_j\right) \neq \left(B_j, C_j\right)$.

So $(B'_1, C'_1) = (B_1, C_1), \cdots, \left(B'_{j-1}, C'_{j-1}\right) = \left(B_{j-1}, C_{j-1}\right)$ and $V'_1 = V_1, \cdots, V'_j = V_j$. Furthermore, the assumption that $j < j_* = min\{j_u, j_v\}$ implies that u and v cannot be in B'_j and C'_j

On the one hand, the fact that $u, v \notin B_j' \cup C_j'$ implies that $B_j' \subseteq V_j$ and $C_j' = \Gamma(B_j') \cap V_j$. So (B_j', C_j') is a pair in G_j which guarantees $\frac{w(C_j')}{w(B_j')} \geq \frac{w(C_j)}{w(B_j)} = \alpha_j$ by Lemma 1. On the other hand since $V_j' = V_j, \Gamma'(B_j) \cap V_j' = C_j$ or $C_j - \{v\}$. The second case happens when $u \in B_j$ and any other vertex in B_j is not incident to v in G_j'. So (B_j, C_j) or $(B_j, C_j - \{v\})$ is a pair in G_j' and $\frac{w(C_j')}{w(B_j')} = \alpha_j' \leq \frac{w(C_j)}{w(B_j)}$. Combining above two inequalities, we know $\frac{w(C_j')}{w(B_j')} = \frac{w(C_j)}{w(B_j)}$. So Proposition 1-(2) tells us that $B_j' \subseteq B_j$, $C_j' \subseteq C_j$ in G_j and $B_j \subseteq B_j'$, $C_j \subseteq C_j'$ in G_j'. Thus $(B_j', C_j') = (B_j, C_j)$which contradicts the assumption.

Item 2 to Item 5 follow from Item 1. For Item 6, if $i_v < j_*$, then $(B_{i_v}', C_{i_v}') = (B_{i_v}, C_{i_v})$ by Item 1. But the fact that v appears in (B_{i_v}, C_{i_v}) ensures that $j_* \leq i_v$ which is a contradiction. Further, Definition 2 promises the correctness of the second inequality of Item 6.■

Here we must note that Lemma 9 does not rule out the possibility that B_j' may contain a C-class vertex, in particular, some a C-class vertex $x \neq v$ in B_j', such as $x = v_7 \neq v = v_5$ illustrated in Figure 2. By the bottleneck decomposition rule, $B_1 = \{v_1, v_2\}$, $C_1 = \{v_3, v_4\}$; $B_2 = \{v_8\}, C_2 = \{v_5, v_9\}$ and $B_3 = \{v_6\}$ and $C_3 = \{v_7\}$. If edge (v_5, v_8) is deleted where $v_5 = v$ and $v_8 = u$, then in the new bottleneck decomposition of G', $B_1' = \{v_1, v_5, v_7\}$. So a C-class vertex $v_7 \neq v$ belongs to B_1'.

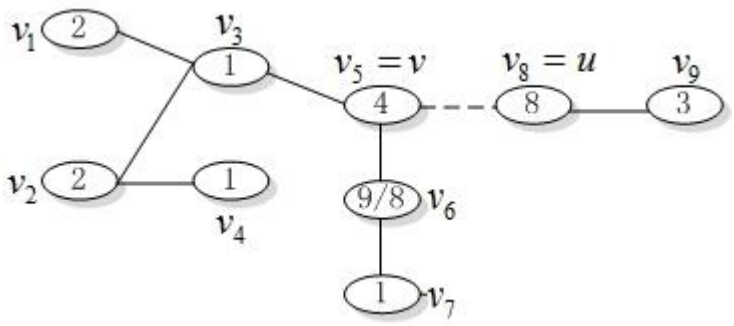

Figure 2: B_1' contains C-class vertex v_7 where the number in circle is the weight of each vertex.

Next we introduce a technique, *Dense Kernel Removal*, for subsequent discussions. The goal is to derive a contradiction to the minimality of the α-ratio of pair (B_i, C_i) in the bottleneck decomposition if we suppose the desired result does not hold.

Definition 3 (*Dense Kernel Removal*). Given $B \subseteq B_i$ and $C \subseteq C_i$ for a pair (B_i, C_i) in the bottleneck decomposition, let $C_i^c = C_i - C$ and $B_i^c = B_i - B$. If $\Gamma(B_i^c) \cap V_i \subseteq C_i^c$ and $\frac{w(C)}{w(B)} > \alpha_i$, then removing the pair (B, C) from (B_i, C_i) would render a pair (B_i^c, C_i^c) such that $\frac{w(C_i^c)}{w(B_i^c)} < \alpha_i$, and hence a contradiction to the definition of the minimum inclusive expansion ratio. We should denote it by $(B_i^c, C_i^c) = DKR(B, C; B_i, C_i)$ for simplicity of presentation.

Lemma 3. Let $\emptyset \neq C \subsetneq C_i$ and $B = \Gamma(C) \cap B_i$, then $\frac{w(B)}{w(C)} \geq \frac{1}{\alpha_i}$.

Proof. Suppose this claim does not hold for some B, C and α_i, then $\frac{w(B)}{w(C)} < \frac{1}{\alpha_i}$, i.e., then $\frac{w(C)}{w(B)} > \alpha_i$, Let $C_i^c = C_i - C$ and $B_i^c = B_i - B$. We have $\alpha_i = \frac{w(C)+w(C_i^c)}{w(B)+w(B_i^c)}$. Therefore $\frac{w(C_i^c)}{w(B_i^c)} < \alpha_i$. It is a case of *Dense Kernel Removal*, denoted as $(B_i^c, C_i^c) = DKR(B, C; B_i, C_i)$.

By definition of B, for any vertex $x \in B_i^c$ we have $\Gamma(x) \cap C = \emptyset$. Therefore $\Gamma(B_i^c) \cap V_i \subseteq C_i^c$ and $\frac{w(\Gamma(B_i^c) \cap V_i)}{w(B_i^c)} \leq \frac{w(C_i^c)}{w(B_i^c)} < \alpha_i$. We find another pair $(B_i^c, \Gamma(B_i^c) \cap V_i)$in G_i such that its α-ratio is strictly less than α_i, a contradiction to B_i being the maximal bottleneck of G_i.

The following are two important results proved by using the dense kernel removal

technique.

Lemma 4 (Key Lemma). Consider the bottleneck decompositions $\mathcal{B}$ and $\mathcal{B}'$ of G and G'. Each of the following conditions implies that for any $1 \le t \le k'$ with $\alpha'_t < 1, B'_t \cap (\bigcup_{i=1}^{k} C_i) = \emptyset$:

1. For the case that $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$, u and v are both in C'-class (Case 1);

2. For the case that $(u, v) \in B_i \times C_i$ with $\alpha_i < 1, i = 1,2,\cdots,k$, v is in C'-class (Case 2), and

3. For the case that $(u, v) \notin B_i \times C_i, i = 1,2,\cdots,k$ (Case 3).

Proof. Let t be the smallest index such that B'_t with $\alpha'_t < 1$ contains a C-class vertex x and suppose $x \in B'_t \cap C_l \subseteq V'_t$ for some $l \in \{1,2,\cdots,k\}$. We derive a contradiction to the minimality of t in the choice of B'_t.

.

First we prove that $\Gamma(x) \cap B_l \subseteq C'_t$. The condition that v and u are in C'-class for Case 1 shows that $x \neq u$ and v because x is B'-class. For Case 2, u is a B-class vertex and v is in C'-class guarantees that $x \neq u$ and v. Therefore $\Gamma(x) = \Gamma'(x)$ in Case 1 and Case 2. As $x \in C_l$, if $x \in \{u, v\}$, $B_l \cap \Gamma(x) \cap \{u, v\} = \emptyset$ in Case 3. We have $\Gamma(x) \cap B_l = \Gamma'(x) \cap B_l$ in all three cases.

If $\Gamma(x) \cap B_l \nsubseteq C'_t$, then there is at least one vertex $y \in \Gamma(x) \cap B_l = \Gamma'(x) \cap B_l$ not in C'_t. As $x \in B'_t$, such a vertex y must be deleted in one of the first $t-1$ steps in the construction of B' and it should be in either B'_h or $C'_h, h \in \{1,2,\cdots,t-1\}$.

If y is in B'_h, then $x \in \Gamma'(y)$must be in C'_h and should have been deleted at step $h' \le h < t$. So x could not be in V'_t and this is a contradiction. If y is in C'_h, then there must be another vertex z such that $z \in B'_h$ and $(z, y) \in E$. The fact $y \in \Gamma(x) \cap B_l$ which means that y is a B-class vertex indicates that z is a C-class

vertex (B-class vertices being independent) or both $l = k$ and $B_l = C_l$ (the only other possibility). In both cases, we have found another B'_h with index $h < t$ which contains a C-class vertex z, a contradiction implying that $\Gamma(x) \cap B_l \subseteq C'_t$ for any vertex $x \in B'_t \cap C_l, l \in \{1,2,\cdots,k\}$.

Second, we partition B'_t (and C'_t, respectively) into two disjoint subsets as follows:

(1)$B'_t = B'_{t1} \cup B'_{t2}$ where $B'_{t1} = \bigcup_{l=1}^{k} \bigcup_{x \in B'_t \cap C_l} \{x\}$; $B'_{t2} = B'_t - B'_{t1}$;

(2)$C'_t = C'_{t1} \cup C'_{t2}$ where $C'_{t1} = \bigcup_{l=1}^{k} \bigcup_{x \in B'_t \cap C_l} (\Gamma(x) \cap B_l)$; $C'_{t2} = C'_t - C'_{t1}$.

Fixing l: $1 \le l \le k$, let $C_{sub} = \bigcup_{x \in B'_t \cap C_l} \{x\} \subseteq C_l$ and $B_{sub} = \bigcup_{x \in B'_t \cap C_l} (\Gamma(x) \cap B_l) = \Gamma(C_{sub}) \cap B_l$. Thus we have $\frac{w(B_{sub})}{w(C_{sub})} \ge \frac{1}{\alpha_l} \ge 1$ by Lemma 10. As $\alpha'_t < 1$, it follows that $\frac{w(C'_{t1})}{w(B'_{t1})} \ge 1 > \alpha'_t$. Therefore $(B^c_{t2}, C^c_{t2}) = DKR(B, C; B'_t, C'_t)$ which in turn implies $\frac{w(C'_{t2})}{w(B'_{t2})} < \alpha'_t$.

Further, by definition, all C-class vertices of B'_t are contained in B'_{t1}.
Therefore B'_{t2} only contains B-class vertices and its neighbors are all C-class.

As all vertices in C'_{t1} are B-class, $(\Gamma(B'_{t2}) \cap V'_t) \cap C'_{t1} = \emptyset$ and $(\Gamma(B'_{t2}) \cap V'_t) \subseteq C'_{t2}$. Then $\frac{w(\Gamma(B'_{t2}) \cap V'_t)}{w(B'_{t2})} \le \frac{w(C'_{t2})}{w(B'_{t2})} < \alpha'_t$

Here another pair $(B'_{t2}, \Gamma(B'_{t2}) \cap V'_t)$ with $B'_{t2} \subseteq V'_t$ is found such that the α-ratio is strictly less than α'_t. Such a contradiction concludes the Key Lemma.■

Key Lemma illustrates that if the conditions in Lemma 3 are satisfied, then there is no C-class vertex in B'-class with $\alpha' < 1$. Symmetrically, we introduce the following Main Lemma.

Lemma 5 (Main Lemma). Consider the bottleneck decompositions $\mathcal{B}$ and $\mathcal{B}'$ of G and G. We have that if the case that $(u, v) \in B_i \times C_i$, and u, v are both in B-class does not happen, then $B_t \cap \left(\bigcup_{\alpha'_l<1} C'_l\right) = \emptyset,\ t = 1,2,\cdots,k$.

Proof. If the claim does not hold, we choose the minimum t such that the above result fails, i.e. $B_t \cap \left(\bigcup_{\alpha'_l<1} C'_l\right) \neq \emptyset$. Thus there is an index l with $\alpha'_l < 1, 1 \leq l \leq k'$, such that $x \in B_t \cap C'_l$.

We claim that $\Gamma'(x) \cap B'_l \subseteq V_t$. Otherwise, let $y \in \Gamma'(x) \cap B'_l - V_t$ be deleted in the first $t-1$ steps. Therefore, $y \in C_h$ with $h < t$; otherwise if $y \in B_h$, then x should be in C_h and be deleted before step t. Hence there exists $z \in B_h$ such that $(z, y) \in B_h \times C_h$. If $(z, y) \in E(G')$, then the fact that $y \in B'_l$ and $\alpha'_l < 1$ implies that z is in C'-class, a contradiction to the choice of t and x. Otherwise if $(z, y) \notin E(G')$, then $z = u$ and $y = v$ by the assumption that v is C-class. So the condition that $(u, v) \in B_i \times C_i$, and u, v are not both in B'-class ensures that z must be in C'-class because $y = v$ is in $\Gamma'(x) \cap B'_l \subseteq B'_l$. So $z \in B_h \cap \left(\bigcup_{\alpha'_l<1} C'_l\right)$ with $h < t$, which is a contradiction to the choice of t and x.

Therefore, for any $x \in B_t \cap C'_l$ with $\alpha'_l < 1$, we have $\Gamma'(x) \cap B'_l \subseteq V_t$. It follows that $\bigcup_{x \in B_t \cap C'_l, \alpha'_l<1} \Gamma'(x) \cap B'_l \subseteq C_t$. Let us partition B_t (and C_t, respectively) into two disjoint subsets $B_{t1} \cup B_{t2}$ (and $C_{t1} \cup C_{t2}$) as follows:

(1)$B_{t1} = \bigcup_{l=1}^{k'} \bigcup_{x \in B_t \cap C'_l, \alpha'_l<1} \{x\}$; $B_{t2} = B_t - B_{t1}$

(2)$C_{t1} = \bigcup_{l=1}^{k'} \bigcup_{x \in B_t \cap C'_l, \alpha'_l<1} (\Gamma'\{x\} \cap B'_l)$; $C_{t2} = C_t - C_{t1}$

Fixing l: we have $C'_{sub} = \bigcup_{x \in B_t \cap C'_l, \alpha'_l<1} \{x\} \subseteq C'_l$ and $B'_{sub} = \bigcup_{x \in B_t \cap C'_l, \alpha'_l<1} (\Gamma'\{x\} \cap B'_l) \subseteq \Gamma'(C'_{sub}) \cap B'_l$. Then we obtain $\frac{w(B'_{sub})}{w(C'_{sub})} \geq \frac{1}{\alpha'_l} > 1$ by Lemma 3. It follows

that $\frac{w(C_{t1})}{w(B_{t1})} > 1 \geq \alpha_t$. Therefore, $(B_{t2}, C_{t2}) = DKR(B_{t1}, C_{t1}; B_t, C_t)$ which in turn implies $\frac{w(C_{t2})}{w(B_{t2})} < \alpha_t$.

In addition, all B'-class vertices of B_t are in B_{t2} and all its neighbors are C'-class. Especially if B_{t2} contains vertex x from $B'_{k'}$, $\alpha'_{k'} = 1$, then x does not has neighbors in B'_l, $\alpha'_l < 1$. We know that all vertices have the same neighborhoods in G and G' except for u and v. Further condition in lemma excludes the case that $u \in B_{t2}, v \in C_{t1}$which leads to $\Gamma(u) \nsubseteq C_{t2}$. So $(\Gamma(B_{t2}) \cap V_t) \cap C_{t1} = \emptyset$ and $(\Gamma(B_{t2}) \cap V_t) \subseteq C_{t2}$. Then $\frac{w(\Gamma(B_{t2}) \cap V_t)}{w(B_{t2})} \leq \frac{w(C_{t2})}{w(B_{t2})} < \alpha_t$.

Here another pair $(B_{t2}, \Gamma(B_{t2}) \cap V_t)$ with $B_{t2} \subseteq V_t$ is found such that its α-ratio is strictly less than α_t. Such a contradiction concludes Lemma 5.■

From Main Lemma, the following corollary can be obtained directly.

Corollary 1. If $u \in B_{i_u}$, then u cannot be in a C'-class with α-ratio < 1. In the following we show that the decomposition of G' is equal to that of G if $(u, v) \notin B_i \times C_i, i = 1,2, \cdots, k$.

Theorem 1. If $(u, v) \notin B_i \times C_i, i = 1,2, \cdots, k$, then $\mathcal{B} = \mathcal{B}'$.

Proof. First we shall prove that $(B_1, C_1) = (B'_1, C'_1)$. Because $(u, v) \notin B_i \times C_i, i = 1,2, \cdots, k$, it is true that $k \geq 2$ or $k = 1$ and $u \in C_1, v \in C_2$ with $\alpha_1 < 1$. If the second claim does not hold, then $(u, v) \in B_1 \times C_1$. It's a contradiction. Thus $\alpha'_1 \leq \alpha_1 < 1$. Further, it is impossible that $u \in B_1$, otherwise $v \in C_1$ which contradicts the condition $(u, v) \notin B_i \times C_i$. By Key Lemma-(3), we know that there is no any C-class vertex in B'_1. Let us and partition B'_1 (and C'_1, respectively) into two disjoint subsets $B'_{11} \cup B'_{12}$ (and $C'_{11} \cup C'_{12}$) as follows:

(1)$B'_{11} = \bigcup_{i=2}^{k}(B'_1 \cap B_i); B'_{12} = B'_1 - B'_{11}$

(2)$C'_{11} = \bigcup_{i=2}^{k}[\Gamma(B'_1 \cap B_i) \cap C_i]; C'_{12} = C'_1 - C'_{11}$

Because $\Gamma'(x) \cap C_i = \Gamma(x) \cap C_i \subseteq \Gamma'(x) \cap V'_1$ for any vertex $x \neq u, v \in B_i, i = 2, \cdots, k$, and $\Gamma(u) \cap C_i \subseteq \Gamma'(u) \cap V'_1$ since $(u, v) \notin B_i \times C_i$, we can get that $\bigcup_{i=2}^{k}[\Gamma(B'_1 \cap B_i) \cap C_i] \subseteq C'_1$ and such a partition is reasonable.

Fixing i: $2 \leq i \leq k$, Lemma 8 ensures that $\frac{w(\Gamma(B'_1 \cap B_i) \cap C_i)}{w(B'_1 \cap B_i)} \geq \alpha_i > \alpha_1 \geq \alpha'_1$. It follows that $\frac{w(C'_{11})}{w(B'_{11})} > \alpha'_1$ if $B'_{11} \neq \emptyset$. Therefore $(B'_{12}, C'_{12}) = DKR(B'_{11}, C'_{11}; B'_1, C'_1)$ which in turn implies that $\frac{w(C'_{12})}{w(B'_{12})} < \alpha'_1$. In addition, all vertices in B'_{12} cannot have neighbors in $C_i, i \geq 2$. So $\Gamma'(B'_{12}) \cap V'_1 \subseteq C'_{12}$. Then we can get a pair $(B'_{12}, \Gamma'(B'_{12}) \cap V'_1)$ in G'_1 whose α-ratio $\frac{w(\Gamma(B'_{12}) \cap V'_1)}{w(B'_{12})} \leq \frac{w(C'_{12})}{w(B'_{12})} < \alpha'_1$.

Such a contradiction indicates that $B'_{11} = \emptyset$ which means that $B'_1 \subseteq B_1$ and $C'_1 \subseteq C_1$. It is easy to check that (B'_1, C'_1) is a pair in G_1 and (B_1, C_1) is a pair in G'_1 since any vertex $x \in B'_1 \subseteq B_1$ cannot be u or v and $\Gamma(x) \cap V_1 = \Gamma'(x) \cap V'_1$. Thus$\alpha'_1 = \alpha_1$ by Lemma 8 and $B'_1 = B_1, C'_1 = C_1$. Moreover $V'_2 = V_2$.

We continue this proof on step $i = 2, \cdots, i_v$. On each step $i, V'_i = V_i$ and $\alpha'_i \leq \alpha_i < 1$. Thus B'_i only contain vertices from B-class by Key Lemma-(3). Especially for step i_v, if $i_v < i_u$ then$\alpha_{i_v} < 1$. If $i_v = i_u$, then $u \in C_{i_u} = C_{i_v}$ with $\alpha_{i_v} < 1$. Otherwise $(u, v) \in B_{i_v} \times C_{i_v}$ with $\alpha_{i_v} = 1$ which contradicts the condition. In both cases, $\alpha'_{i_v} < 1$ and $V'_{i_v} = V_{i_v}$. So we can get $(B'_i, C'_i) = (B_i, C_i)$ and $V'_i + 1 = V_i + 1, i = 2, \cdots, i_v$, by the same method.

Furthermore, the fact that v is deleted on step i_v leads to all edges adjacent to v are removed. So subgraphs $G'_i = G_i, i = i_v + 1, \cdots, k$, and we can easily get

that $(B_i', C_i') = (B_i, C_i), i = i_v + 1, \cdots, k.$ ■

We should emphasize that $(u, v) \notin B_i \times C_i$ includes two cases: Case $u \in B_{i_u}, v \in C_{i_v}$ with $i_v < i_u$ and Case $u \in C_{i_u}, v \in C_{i_v}$ with $i_v \leq i_u$.

3.3 Incentive Properties

In this subsection we propose the Strategy-proof properties of agent u and v by analyzing different cases. It is easy to compute that there are totally 16 cases because agent u can be in $B_{i_u}, C_{i_u}, B'_{j_u}$ *or* C'_{j_u} and v can be in $B_{i_v}, C_{i_v}, B'_{j_v}$ *or* C'_{j_v}. But according to the assumption that u and v are in B-class and C-class respectively if edge (u, v) is between different classes, it is not necessary to consider such cases in which $u \in C_{i_u}$ and $v \in B_{i_v}$. In addition, for the cases in which $u \in B_{i_u}$ and $v \in B_{i_v}$, it is only possible that $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$. Thus u and v have the same statuses in G and Case $u \in B_{i_u}, v \in B_{i_v}, u \in B'_{j_u}, v \in C'_{j_v}$ is equivalent to Case $u \in B_{i_u}, v \in B_{i_v}, u \in C'_{j_u}, v \in B'_{j_v}$ It is enough to discuss one of them. Here we should point out that when such cases in which $u \in C_{i_u}$ or $v \in C_{i_v}$ ($u \in C'_{j_u}$ or $v \in C'_{j_v}$, respectively) are discussed, we mean its α-ratio (α'-ratio) less than 1. Because if its α-ratio (α'-ratio) equals to 1, such cases would come down to Case $u \in B_k$ or $v \in B_k$ with $\alpha_k = 1$ (Case $u \in B'_{k'}$ or $v \in B'_{k'}$ with $\alpha' = 1$, respectively). In the following we will try our best to prove that u and v has no incentive to cheat by removing edge (u, v) in the rest cases.

First we know that if $u \in C_{i_u}, v \in C_{i_v}$ and $u \in B_{i_u}, v \in C_{i_v}$ with $i_v < i_u$, then $(u, v) \notin B_i \times C_i$ and $\mathcal{B}' = \mathcal{B}$ by Theorem 3. So $U'_u = U_u$ and $U'_v = U_v$ for such cases in which

that $u \in C_{i_u}, v \in C_{i_v}$ and $u \in B_{i_u}, v \in C_{i_v}$ with $i_v < i_u$.

We also get that any agent has its utility larger than or equal to its bandwidth if it is in C-class (or in C'-class) and has its utility smaller than or equal to its bandwidth if it is in B-class (or in B'-class) by Proposition 3.

Furthermore, if u and v are both in B-class, it is only possible that $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$ which implies that $U_u = w_u$ and $U_v = w_v$. So $U'_u \leq U_u$ and $U'_v \leq U_v$ for Case $u \in B_k, v \in B_k$ with $\alpha_k = 1$ and $u \in B'_{j_u}, v \in B'_{j_v}$.

Further Corollary 1 ensures the following cases cannot exist because u is in B-class and $u \in C'_{j_u}$ with $\alpha_0 < 1$:

(1) Case $u \in B_k, v \in B_k$ with $\alpha_k = 1, u \in C'_{j_u}, v \in C'_{j_v}$;

(2) Case $u \in B_k, v \in B_k$ with $\alpha_k = 1, u \in C'_{j_u}, v \in B'_{j_v}$;

(3) Case $u \in B_i, v \in C_i, u \in C'_{j_u}, v \in C'_{j_v}$;

(4) Case $u \in B_i, v \in C_i, u \in C'_{j_u}, v \in B'_{j_v}$.

Next, we shall prove our main result for the rest of 2 cases in the following lemmas.

Lemma 6. If $u \in B_i, v \in C_i, u \in B'_{j_u}$ and $v \in B'_{j_v}$, then u and v both have no incentive to cheat.

Proof. Because $U'_v = \alpha'_{j_v} w_v \leq \frac{w_v}{\alpha_{i_v}} = U_v$, agent v has no incentive to cheat. For agent u if $j_u \leq j_v$, then $j_u = j_* \leq i$ by Basic Lemma-(6). Because Basic Lemma-(3) shows that $\alpha'_{j_*} \leq \alpha_{j_*}$ and $j_* \leq i$ which implies $\alpha_{j_*} \leq \alpha_i$, we have $\alpha'_{j_*} \leq \alpha_i$ and $U'_u = \alpha'_{j_*} w_u \leq \alpha_i w_u = U_u$. Thus agent u cannot get benefit by cheating in this case.

It is more complicated to discuss the utility of agent u in G' if $j_v < j_u$. Our main idea is to construct a pair containing u in V'_{j_u} whose α-ratio is smaller than α_i. Thus we can get that $\alpha'_{j_u} < \alpha_i$ and $U'_u < U_u$. In order to obtain this result, the following results are necessary and their proofs are shown in the next subsection.

Fact 1. For each $1 \le l < i, C_l \cap B'_j = \emptyset$ and $B_l \cap C'_j = \emptyset, j = j_v, \cdots, j_u,$ if $(u, v) \in B_i \times C_i$ with $\alpha_i < 1$ and $j_v < j_u$.

Fact 1 shows that no vertex in C_l belongs to B'_j and no vertex in B_l belongs to C'_j, for $1 \le l \le i-1$ and $j_v \le j \le j_u$. Therefore, we can define $B_l^h = B_l - \bigcup_{j=j_v}^{h} B'_j, C_l^h = C_l - \bigcup_{j=j_v}^{h} C'_j, h = j_v, \cdots, j_u - 1$.

For each $h = j_v, \cdots, j_u - 1$, if let $B = \bigcup_{j=j_v}^{h}(B'_j \cap B_l)$ and $C = \bigcup_{j=j_v}^{h}(C'_j \cup C_l)$, then $\Gamma(B) \cap C_l = C$. So if let $B_l^h = B^c = B_l - B$ and $C_l^h = C^c = C_l - C$, then Lemma 1 tells us, for $l = j_v, \cdots, i-1, h = j_v, \cdots, j_u - 1$,

$$\frac{w(C_l^h)}{w(B_l^h)} = \frac{w(C^c)}{w(B^c)} \le \alpha_l < \alpha_i. \quad (1)$$

But if $l = i$, there is no such a nice result in Fact 1. Thus for $h = j_v, \cdots, j_u - 1$, we define $B_i^h = B_i - \bigcup_{j=j_v}^{h}(B'_j \cup C'_j), C_i^h = C_i - \bigcup_{j=j_v}^{h}(B'_j \cup C'_j)$, and have:

Fact 2. If $(u, v) \in B_i \times C_i$ with $\alpha_i < 1$ and $j_v < j_u$, then $\frac{w(C_i^h)}{w(B_i^h)} < \alpha_i, h = j_v, \cdots, j_u - 1$.

Now let us focus on V'_{j_u}. Since $B_l^{j_u-1} \subseteq V'_{j_u}$ and $C_l^{j_u-1} \subseteq V'_{j_u}, l = j_v, \cdots, i$, we define $\hat{B} = \bigcup_{l=j_v}^{i} B_l^{j_u-1}$ and $\hat{C} = \bigcup_{l=j_v}^{i} C_l^{j_u-1}$. Then (1) and Fact 2 tells $\frac{w(\hat{C})}{w(\hat{B})} =$

$\frac{\sum_{l=j_v}^{i} w(C_l^{j_u-1})}{\sum_{l=j_v}^{i} w(B_l^{j_u-1})} < \alpha_i$.

Further, all vertices in $B_l^{j_u-1}$ only can be adjacent to vertices in $C_h^{j_u}, h = j_v, \cdots, l$, in V'_{j_u}. Thus $\Gamma'(\mathrm{B}_{\mathrm{l}}^{\mathrm{j_u-1}}) \cap \mathrm{V}'_{\mathrm{j_u}} \subseteq \hat{C}$ for each $l = j_v, \cdots, i$.

So $\Gamma'(\hat{B}) \cap V'_{j_u} = C$ and $(\hat{B}, \hat{C})$ is a pair in V'_{j_u} whose α-ratio is less than α_i.

Thus $\alpha'_{j_u} \leq \frac{w(\hat{C})}{w(\hat{B})} < \alpha_i$. ■

Lemma 7. If $u \in B_i, v \in C_i, u \in B'_{j_u}$ and $v \in C'_{j_v}$, then u and v both have no incentive to cheat.

Proof. First we shall claim that if $j_v \leq j_u$, then it must be that $j_u = j_v = i$ and $(B_i, C_i) = (B'_i, C'_i)$. On the one hand, Basic Lemma-(3) ensures that $\alpha'_{j_v} \leq \alpha_{j_v}$ because $V'_{j_v} = V_{j_v}$. On the other hand by Basic Lemma-(6), it is true that $j_v \leq i$ if $j_v \leq j_u$. So at step j_v of the decomposition in G, if $u \notin B'_{j_v}$, then all vertices in B'_{j_v} have the same neighborhoods in G and G' which means that $\Gamma(B'_{j_v}) \cap V_{j_v} = C'_{j_v}$. If $u \in B'_{j_v}$, then we still have $\Gamma(\mathrm{u}) \cap \mathrm{V_{j_v}} = (\Gamma'(\mathrm{u}) \cap \mathrm{V_{j_v}}) \cup \{\mathrm{v}\} \subseteq \mathrm{C'_{j_v}}$ as $v \in C'_{j_v}$ and $\mathrm{v} \notin \Gamma'(\mathrm{u})$. Thus (B'_{j_v}, C'_{j_v}) is a pair in V_{j_v}. It follows that $\alpha'_{j_v} \geq \alpha_{j_v}$. At the same time, Basic Lemma-(3) ensures that $\alpha'_{j_v} \leq \alpha_{j_v}$ because $V'_{j_v} = V_{j_v}$. Therefore $\alpha'_{j_v} = \alpha_{j_v}$ and $B'_{j_v} \subseteq B_{j_v}, C'_{j_v} \subseteq C_{j_v}$. Further the condition $v \in C'_{j_v} \subseteq C_{j_v}$ promises that $v \in C_{j_v}$ which means that $j_v = i$. So at step $i = j_v$ of the decomposition in G', (B_i, C_i) is a pair in V'_i. The result $\alpha_i = \alpha'_i$ shows that $B_i \subseteq B'_i$ and $C_i \subseteq C'_i$. Combining the previous result, we have $(B_i, C_i) = (B'_i, C'_i)$ and $j_v = j_u = i$. Under this case, $U'_u = U_u$ and $U'_v = U_v$.

Next let us discuss the case that $j_u < j_v$. Thus $U'_u \leq U_u$ since $j_u \leq i$ and $\alpha'_{j_u} \leq \alpha_i$. But it's much more difficult to analyze the characterization of agent v. In order to get the ideal result, the following fact is necessary:

Fact 3. If $(u, v) \in B_i \times C_i$ and $j_v > j_u, v \in C'_{j_v}$, then there is an index $j_u < j \leq j_v$ satisfying:

(1)$B_h \cap V'_j = \emptyset$ and $C_h \cap V'_j = \emptyset, h = 1,2, \cdots, i-1$;

(2)$B_h \subseteq V'_j$ and $C_h \subseteq V'_j, h = i+1, \cdots, k$;

(3)$\alpha'_j \geq \alpha_i$;

(4)$v \in V'_j$.

The proof of Fact 3 is presented in the next subsection. Fact 3 indicates that V'_j does not have any vertex from $\bigcup_{h=1}^{i-1}(B_h \cup C_h)$, all vertices in $\bigcup_{h=i+1}^{k}(B_h \cup C_h)$ are contained in V'_j and some vertices including v in $B_i \cup C_i$ belong to V'_j. Based on the result of Fact 3-(3), we know that $\alpha'_{j_v} \geq \alpha'_j \geq \alpha_i$ since $j_v \geq j$. Thus $U'_v = \frac{w_v}{\alpha'_{j_v}} \leq \frac{w_v}{\alpha_i} = U_v$. ■

Up to now, all the 11 cases have been analyzed one by one and the following table is a simple summary for this.

Table 3.1 Summary of Analysis for All cases

	$u \in G$	$v \in G$	$u \in G'$	$u \in G'$	Reason for Impossibility / Treatment
1	B_{i_u}	B_{i_v}	B'_{j_u}	B'_{j_v}	$U'_u \leq U_u, U'_v \leq U_v$
2	B_{i_u}	B_{i_v}	C'_{j_u}	B'_{j_v}	Cannot exist by Corollary 1
3	B_{i_u}	B_{i_v}	C'_{j_u}	C'_{j_v}	Cannot exist by Corollary 1

4	C_{i_u}	C_{i_v}	B'_{j_u}	B'_{j_v}	Cannot exist by Theorem 1
5	C_{i_u}	C_{i_v}	C'_{j_u}	B'_{j_v}	Cannot exist by Theorem 1
6	C_{i_u}	C_{i_v}	B'_{j_u}	C'_{j_v}	Cannot exist by Theorem 1
7	C_{i_u}	C_{i_v}	C'_{j_u}	C'_{j_v}	$(B_{i_u}, C_{i_u}) = (B'_{i_u}, C'_{i_u}), (B_{i_v}, C_{i_v}) = (B'_{i_v}, C'_{i_u})$ by theorem 1
8	B_{i_u}	C_{i_v}	C'_{j_u}	C'_{j_v}	Cannot exist by Corollary 1
9	B_{i_u}	C_{i_v}	C'_{j_u}	B'_{j_v}	Cannot exist by Corollary 1
10	B_{i_u}	C_{i_v}	B'_{j_u}	B'_{j_v}	Cannot exist by Theorem 1 if $i_u > i_v$ Dealt with in Lemma 6 if $i_u = i_v$
11	B_{i_u}	C_{i_v}	B'_{j_u}	C'_{j_v}	$(B_{i_u}, C_{i_u}) = (B'_{i_u}, C'_{i_u}), (B_{i_v}, C_{i_v}) = (B'_{i_v}, C'_{i_u})$ by theorem 1if $i_u > i_v$ Dealt with in Lemma 7 if $i_u = i_v$

Based on the analysis above, the main result in the following can be induced directly, except the proof for those three fact which will be showed in the next subsection.

Theorem 2. (Main Result) Given the bandwidth allocation mechanism obtained by the market equilibrium from the bottleneck decomposition, no agent has incentive to cheat by cutting off any of its incident edges.

3.4 The Proofs of Facts 1, 2 and 3

Fact 1. For each $1 \leq l < i, C_l \cap B'_j = \emptyset$ and $B_l \cap C'_j = \emptyset, j = j_v, \cdots, j_u,$ if $(u, v) \in B_i \times C_i$ with $\alpha_i < 1$ and $j_v < j_u$.

Proof. We first show the correctness of the claim that $B'_j \cap C_l = \emptyset$ for each $j = j_v, \cdots, j_u$ and $l = 1, \cdots, i-1$. Obviously this claim holds for $l = 1, \cdots, j_v - 1$, since $V_{j_v} = V'_{j_v}$.

Now we shall prove this statement for $l = j_v, \cdots, i-1$ by mathematical induction method. When $j = j_v$, it is clear that $V'_{j_v} = V_{j_v}$ and for any $l \geq j_v$, $B_l \subseteq V'_{j_v}$ and $C_l \subseteq V'_{j_v}$. Especially for $j_v \leq l < i$, the neighborhood in B_l of any subset of C_l does not change, that is $\Gamma'\left(B'_{j_v} \cap C_l\right) \cap B_l = \Gamma\left(B'_{j_v} \cap C_l\right) \cap B_l \subseteq C'_{j_v}$, because u and v are not in $B_l \cup C_l, j_v \leq l < i$. Suppose this claim does not hold for $j = j_v$, then B'_{j_v} (and C'_{j_v}, respectively) can be partitioned into two disjoint subset $B'_{j_v 1} \cup B'_{j_v 2}$ (and $C'_{j_v 1} \cup C'_{j_v 2}$, respectively) as follows:

(1)$B'_{j_v 1} = \bigcup_{l=j_v}^{i-1}(B'_{j_v} \cap C_l)$, $B'_{j_v 2} = B'_{j_v} - B'_{j_v 1}$;

(2)$C'_{j_v 1} = \bigcup_{l=j_v}^{i-1}(\Gamma\left(B'_{j_v} \cap C_l\right) \cap B_l)$, $C'_{j_v 2} = C'_{j_v} - C'_{j_v 1}$.

Fixing l: $j_v \leq l \leq i-1$, let $C_{sub} = B'_{j_v} \cap C_l \subseteq C_l$ and $B_{sub} = \Gamma(C_{sub}) \cap B_l$.

Thus $\frac{w(B_{sub})}{w(C_{sub})} \geq \frac{1}{\alpha_l} > 1 \geq \alpha'_{j_v}$ by Lemma 3. Therefore,

$(B'_{j_v 2}, C'_{j_v 2}) = DKR(B'_{j_v 1}, C'_{j_v 1}; B'_{j_v}, C'_{j_v})$ which in turn implies $\frac{w\left(C'_{j_v 2}\right)}{w\left(B'_{j_v 2}\right)} < \alpha'_{j_v}$.

Also we know that there is no edge between $B'_{j_v 2}$ and $C'_{j_v 1}$. Thus $B'_{j_v 2}$'s neighborhood in G'_{j_v} is contained in $C'_{j_v 2}$, that is $\Gamma'\left(B'_{j_v 2}\right) \cap V'_{j_v} \subseteq C'_{j_v 2}$. So we can get a new pair $(B'_{j_v 2}, \Gamma'\left(B'_{j_v 2}\right) \cap V'_{j_v})$ with a smaller α-ratio. It's a contradiction and this claim holds for $j = j_v$.

If this claim is true for $j = j_v + 1, \cdots, j_u - 1$, then let us discuss the case that $j = j_u$.

Suppose to the contrary that $x \in B'_{j_u} \cap C_l \neq \emptyset, l = j_v, \cdots, i-1$. Then $\Gamma(x) \cap B_l \subseteq V'_{j_u}$. Otherwise at least one vertex $y \in \Gamma(x) \cap B_l$ should be deleted before step j_u. Since all $B'_j, j = j_v, \cdots, j_u - 1$, only contain vertices from $C_h, h \geq i$, and $B_h, h \geq j_v$, there is no any neighbor of y in B'_j. Thus such a vertex y cannot be a C'-class vertex and it only can be in some B'_j. Thus under this situation, as a neighbor of y, vertex x should be deleted at step $j < j_u$ while not at step j_u. It's a contradiction.

Since $\Gamma(x) \cap B_l \subseteq V'_{j_u}$ and $\frac{w(\Gamma(x) \cap B_l)}{w_x} > 1$, we can remove all such vertices from B'_{j_u} and their neighbors in B_l from C'_{j_u} to get a new pair with a smaller α-ratio. So this contradiction shows this claim is correct.

Further by applying the first claim, we can induce that $B_l \cap C'_j = \emptyset, 1 \leq l \leq i-1$ and $j_v \leq j \leq j_u$. Otherwise if $x \in B_l \cap C'_j \neq \emptyset$, then there must be a neighbor y of x in B'_j. Since $x \in B_l$, all its neighbors only can be from $C_h, h \leq l$. So $y \in C_h \cap B'_j \neq \emptyset, 1 \leq h \leq i-1$ and $j_v \leq j \leq j_u$, which contradicts the first claim.■

Fact 2. If $(u, v) \in B_i \times C_i$ with $\alpha_i < 1$ and $j_v < j_u$, then $\frac{w(C_i^h)}{w(B_i^h)} < \alpha_i, h = j_v, \cdots, j_u - 1$.

Proof. We shall prove this claim by the Mathematical Induction Method. First at step j_v of the decomposition in G', $V'_{j_v} = V_{j_v}, \alpha'_{j_v} \leq \alpha_{j_v}$ by Basic Lemma and B'_{j_v} may have vertices from $B_l, l \geq j_v, C_h, h \geq i$, by Fact 1. Thus B'_{j_v} (and C'_{j_v}, respectively) can be partitioned into two disjoint subset $B'_{j_v 1} \cup B'_{j_v 2}$ (and $C'_{j_v 1} \cup C'_{j_v 2}$, respectively) as follows:

(1)$B'_{j_v1} = B'_{j_v} \cap C_i, B'_{j_v2} = B'_{j_v} - B'_{j_v1} = [\cup_{l=j_v}^{k}(B'_{j_v} \cap B_l)] \cup [\cup_{h=i+1}^{k}(B'_{j_v} \cap C_h)]$;

(2)$C'_{j_v1} = \Gamma'^{\left(B'_{j_v} \cap C_i\right)} \cap V_{j_v}, C'_{j_v2} = C'_{j_v} - C'_{j_v1}$.

Since $j_v < j_u \leq k', \alpha'_{j_v} < 1$. Further $C'_{j_v1} = \Gamma'\left(B'_{j_v} \cap C_i\right) \cap V'_{j_v} = \Gamma\left(B'_{j_v} \cap C_i\right) \cap B_i - \{u\}$ because $v \in B'_{j_v1}$. There is no any edge between B_l and $B_i, l \geq j_v$, and no any edge between C_h and $B_i, h \geq i+1$. Thus $\Gamma'\left(B'_{j_v2}\right) \cap V'_{j_v} \subseteq C'_{j_v2}$. It follows that $\frac{w\left(C'_{j_v2}\right)}{w\left(B'_{j_v2}\right)} \geq \frac{w\left(\Gamma'\left(B'_{j_v2}\right) \cap V'_{j_v}\right)}{w\left(B'_{j_v2}\right)} \geq \alpha'_{j_v}$, as $(B'_{j_v2}, \Gamma'\left(B'_{j_v2}\right) \cap V'_{j_v})$ is a pair in G'_{j_v}.

Therefore $(B'_{j_v1}, C'_{j_v1}) = DKR(B'_{j_v2}, C'_{j_v2}; B'_{j_v}, C'_{j_v})$ which in turn implies that $\frac{w\left(C'_{j_v1}\right)}{w\left(B'_{j_v1}\right)} \leq \alpha'_{j_v} < 1$.

Now let us focus on B_i and C_i. Similarly B_i (and C_i, respectively) can be partitioned into three disjoint subset $B_{i1} \cup \backslash B_{i2} \cup B_{i3}$ (and $C_{i1} \cup C_{i2} \cup C_{i3}$, respectively) as follow:

(1)$B_{i1} = B'_{j_v} \cap B_i, B_{i2} = \Gamma(C_{i2}) \cap B_i - \{u\}, B_{i3} = B_i - B_{i1} - B_{i2}$;

(2)$C_{i1} = \Gamma(B_{i1}) \cap C_i, C_{i2} = B'_{j_v} \cap C_i, C_{i3} = C_i - C_{i1} - C_{i2}$.

It is obvious that $v \in C_{i2} = B'_{j_v1}$ and $B_{i2} = C'_{j_v1}$. Thus $\frac{w(B_{i2})}{w(C_{i2})} = \frac{w\left(C'_{j_v1}\right)}{w\left(B'_{j_v1}\right)} \leq \alpha'_{j_v} < 1$ which implies that $\frac{w(C_{i2})}{w(B_{i2})} \geq \frac{1}{\alpha'_{j_v}} > 1 \geq \alpha_i$. In addition combining the fact that $\frac{w(C_{i1})}{w(B_{i1})} \geq \alpha_i$, we have $\frac{w(C_{i3})}{w(B_{i3})} < \alpha_i$.

Because $B_{i3} = B_i - (B'_{j_v} \cup C'_{j_v}) = B_i^{j_v}$ and $C_{i3} = C_i - (B'_{j_v} \cup C'_{j_v}) = C_i^{j_v}$, this claim holds for $h = j_v$.

Suppose this claim is right for $h = j_v + 1, \cdots, j-1, j \leq j_u - 1$. Let us consider

step j of the decomposition in G'. Since $j < j_u \leq k'$, we have $\alpha'_j < 1$.

Similarly B'_j and C'_j can be partitioned as:

(1)$B'_{j1} = B'_j \cap C_i^{j-1}, B'_{j2} = B'_j - B'_{j1} = \left[\cup_{l=j_v}^{k}\left(B'_j \cap B_l^{j-1}\right)\right] \cup [\cup_{h=i+1}^{k}(B'_j \cap C_h^{j-1})]$;

(2)$C'_{j1} = \Gamma'\left(B'_j \cap C_i^{j-1}\right) \cap V'_j, C'_{j2} = C'_j - C'_{j1}$.

By the same reason, we know $\Gamma'\left(B'_{j2}\right) \cap V'_j \subseteq C'_{j2}$. So $\frac{w\left(C'_{j2}\right)}{w\left(B'_{j2}\right)} \geq \frac{w\left(\Gamma'\left(B'_{j2}\right) \cap V'_j\right)}{w\left(B'_{j2}\right)} \geq \alpha'_j$,

as $(B'_{j2}, \Gamma'\left(B'_{j2}\right) \cap V'_j)$ is a pair in V'_j. Therefore $(B'_{j1}, C'_{j1}) =$

$DKR(B_{j2}, C'_{j2}; B'_j, C'_j)$ which in turn implies that $\frac{w\left(C'_{j1}\right)}{w\left(B'_{j1}\right)} \leq \alpha'_j < 1$.

Now consider the partitions of B_i^{j-1} and C_i^{j-1}:

(1)$B_{i1}^{j-1} = B'_j \cap B_i^{j-1}, B_{i2}^{j-1} = \Gamma(C_{i2}) \cap B_i^{j-1} - \{u\}, B_{i3}^{j-1} = B_i^{j-1} - B_{i1}^{j-1} - B_{i2}^{j-1}$;

(2)$C_{i1}^{j-1} = \Gamma\left(B_{i1}^{j-1}\right) \cap C_i^{j-1}, C_{i2}^{j-1} = B'_j \cap C_i^{j-1}, C_{i3}^{j-1} = C_i^{j-1} - C_{i1}^{j-1} - C_{i2}^{j-1}$.

It is obvious that $C_{i2}^{j-1} = B'_{j1}$ and $B_{i2}^{j-1} = C'_{j1}$. Thus $\frac{w\left(B_{i2}^{j-1}\right)}{w\left(C_{i2}^{j-1}\right)} = \frac{w\left(C'_{j1}\right)}{w\left(B'_{j1}\right)} \leq \alpha'_j < 1$ which

implies that $\frac{w\left(C_{i2}^{j-1}\right)}{w\left(B_{i2}^{j-1}\right)} \geq \frac{1}{\alpha'_j} > 1 \geq \alpha_i > \frac{w\left(C_i^{j-1}\right)}{w\left(B_i^{j-1}\right)}$. In addition, $\frac{w\left(C_{i1}^{j-1}\right)}{w\left(B_{i1}^{j-1}\right)} \geq$

$\frac{w\left(C_i^{j-1}\right)}{w\left(B_i^{j-1}\right)}$ otherwise we can construct another pair in G_i whose α-ratio is strictly less

than α_i. Therefore $(B_{i3}^{j-1}, C_{i3}^{j-1}) = DKR(B_{i1}^{j-1} \cup B_{i2}^{j-1}, C_{i1}^{j-1} \cup$

$C_{i2}^{j-1}; B_i^{j-1}, C_i^{j-1})$ which in turn implies that $\frac{w\left(C_{i3}^{j-1}\right)}{w\left(B_{i3}^{j-1}\right)} < \frac{w\left(C_i^{j-1}\right)}{w\left(B_i^{j-1}\right)} < \alpha_i$. Because $B_{i3}^{j-1} =$

B_i^j and $C_{i3}^{j-1} = C_i^j$, this claim holds for $h = j$.

So by the Mathematical Induction Method, Fact 2 holds. ■

Fact 3. If $(u,v) \in B_i \times C_i$ and $j_v > j_u, v \in C'_{j_v}$, then there is an index $j_u < j \leq j_v$ satisfying:

(1)$B_h \cap V'_j = \emptyset$ and $C_h \cap V'_j = \emptyset, h = 1,2,\cdots,i-1$;

(2)$B_h \subseteq V'_j$ and $C_h \subseteq V'_j, h = i+1,\cdots,k$;

(3)$\alpha'_j \geq \alpha_i$;

(4)$v \in V'_j$.

Proof. Since $(u,v) \in B_i \times C_i, \alpha_i < 1$ and v is in C'-class, each B'_j with $\alpha'_j < 1$ only contain B-class vertices by Key Lemma-(2). First let us consider step j_u of the decomposition in G'. Because $j_u = j_* \leq i, V_{j_u} = V'_{j_u}, B_l \subseteq V'_{j_u}$ and $C_l \subseteq V'_{j_u}, l = j_u,\cdots,k$. Further $\alpha'_{j_u} < \alpha_i$. Otherwise $j_u = i$ and $(B_i, C_i) = (B'_i, C'_i)$ which means that $j_v = j_u$. It's a contradiction. Based on the fact that $\alpha'_{j_u} < \alpha_i, B'_{j_u}$ and C'_{j_u} can be proved that it does not contain any vertex from B_l and C_l for $l = i+1,\cdots,k$, by using the *Dense Kernel Removal* technique. If not, B'_{j_u} (and C'_{j_u}, respectively) can be partitioned in a similar way as before:

(1)$B'_{j_u 1} = \cup_{l=i+1}^{k}(B'_{j_u} \cap B_l), B'_{j_u 2} = B'_{j_u} - B'_{j_u 1}$;

(2)$C'_{j_u 1} = \cup_{l=i+1}^{k}(\Gamma'(B'_{j_u} \cap B_l) \cap C_l), C'_{j_u 2} = C'_{j_u} - C_{j_u 1}{}'$.

Because $u \notin B_l, l = i+1,\cdots,k$, and $C_l \subseteq V'_{j_u}, \Gamma'(B'_{j_u} \cap B_l) \cap C_l = \Gamma(B'_{j_u} \cap B_l) \cap C_l$.

Thus $\frac{w(\Gamma(B'_{j_u} \cap B_l) \cap C_l)}{w(B'_{j_u} \cap B_l)} \geq \alpha_l > \alpha_i$ by Lemma 1 which leads to $\frac{w(C'_{j_u 1})}{w(B'_{j_u 1})} > \alpha_i > \alpha'_{j_u}$ and $(B'_{j_u 2}, C'_{j_u 2}) = DKR(B'_{j_u 1}, C'_{j_u 1};\ B'_{j_u}, C'_{j_u})$ with $\frac{w(C'_{j_u 1})}{w(B'_{j_u 1})} < \alpha'_{j_u}$. We also know

that all vertices in $B'_{j_u 2}$ are from $B_l, l = j_u, \cdots, i$, as B'_{j_u} only contains B'-class vertex. Moreover, there is no edge between $B'_{j_u 2}$ and $C_l, l > i$. So $\Gamma'(B'_{j_u 2}) \cap V'_{j_u} \subseteq C'_{j_u 2}$. There is a pair $(B'_{j_u 2}, \Gamma'(B'_{j_u 2}) \cap V'_{j_u})$ in G'_{j_u} whose α-ratio is less than α'_{j_u}. It's a contradiction. So $B_l \subseteq V'_{j_u+1}$ and $C_l \subseteq V'_{j_u+1}, l = i+1, \cdots, k$.

In addition, if we let $B'_{sub} = B'_{j_u} \cap B_i$ and $C'_{sub} = \Gamma'(B'_{j_u} \cap B_i) \cap C_i \subseteq C'_{j_u}$, then $B'_{sub} \neq \emptyset$ since $u \in B'_{j_u} \cap B_i$. We can conclude that $\frac{w(C'_{sub})}{w(B'_{sub})} < \alpha_i$ which indicates that $\Gamma'(B'_{j_u} \cap B_i) \cap C_i = \Gamma(B'_{j_u} \cap B_i) \cap C_i - \{v\}$. Otherwise if $\frac{w(C'_{sub})}{w(B'_{sub})} \geq \alpha_i$, then we also can apply the *Dense Kernel Removal* technique to obtain a new pair whose α-ratio is less than α'_{j_u} by excluding B'_{sub} and C'_{sub} from B'_{j_u} and C'_{j_u}, respectively. It's a contradiction. So $v \notin C'_{j_u}$ and must be in V'_{j_u+1}.

Continue the decomposition process in G', there are some vertices in $B_l, l = j_u, \cdots, i$, being deleted. Consider at any step j, if $V'_j \cap B_l = \emptyset$ for each $l < i$ and $\alpha'_j \geq \alpha_i$, then j just is the index what we want and this claim holds. Otherwise, $\alpha'_j < \alpha_i$ and $V'_j \cap B_l = \emptyset$ for each $l < i$ or $V'_j \cap B_l \neq \emptyset$ for some $l < i$. We only need to show that $j < j_v$ for both of those two cases and we can still continue the decomposition process until at a new step $j', B'_{j'} \cap B_l = \emptyset$ for each $l < i$ and $\alpha'_{j'} \geq \alpha_i$, then j' will be the desired index.

To prove the conclusion, we first claim that $\alpha'_j < \alpha_i$ for both cases. Obviously, it is true for the first case. For the second case, let $t < i$ be the smallest index such that $V'_j \cap B_t \neq \emptyset$. So B_t and C_t can be partitioned into:

(1)$B_{t1} = \bigcup_{h=j_u}^{j-1} (B'_h \cap B_t)$, $B_{t2} = B_t - B_{t1} = B_t \cap V'_j$;

(2)$C_{t1} = \bigcup_{h=j_u}^{j-1}(\Gamma'(B'_h \cap B_t) \cap V'_h)$, $C_{t2} = C_t - C_{t1}$.

Because all vertices in B_t have the same neighborhoods in G and G', $\Gamma(B_{t1}) \cap C_t = C_{t1}$. Lemma 1 ensures that $\frac{w(C_{t2})}{w(B_{t2})} \leq \alpha_t < \alpha_i$ since $\frac{w(C_{t1})}{w(B_{t1})} \geq \alpha_t$. The smallest index guarantees that $B_l \cap V'_j = \emptyset$ and $C_l \cap V'_j = \emptyset, l < t$. So the neighbors of each vertex in B_{t2} which are restricted in V'_j must be in C_{t2} and we find a pair (B_{t2}, C_{t2}) in V'_j whose α-ratio is less than α_i which implies that $\alpha'_j < \alpha_i$.

For both two cases, we also can prove that $B_l \cap B'_j = \emptyset$ and $C_l \cap C'_j = \emptyset, l = i + 1, \cdots, k$, by using the result that $\alpha'_j < \alpha_i$ and the *Dense Kernel Removal* technique.

Thus $B_l \subseteq V'_j$ and $C_l \subseteq V'_j, l = i + 1, \cdots, k$. Similarly $\frac{w\left(\Gamma'\left(B'_j \cap B_i\right) \cap C_i\right)}{w\left(B'_j \cap B_i\right)} < \alpha_i$ if $B'_j \cap B_i \neq \emptyset$. Otherwise, we can obtain a new pair with smaller α-ratio by excluding $B'_j \cap B_i$ and $\Gamma'\left(B'_j \cap B_i\right) \cap C_l$ from B'_j and C'_j, respectively. In order to show $v \notin C'_j$, we shall let $B_{sub} = \bigcup_{h=j_u}^{j}(B'_h \cap B_i)$, $C_{sub} = \bigcup_{h=j_u}^{j}(\Gamma'(B'_h \cap B_i) \cap V'_h)$. So $\frac{w(C_{sub})}{w(B_{sub})} = \frac{\sum_{h=j_u}^{j} w(\Gamma'(B'_h \cap B_i) \cap C_l)}{\sum_{h=j_u}^{j} w(B'_h \cap B_i)} < \alpha_i$. But if $v \in C'_j$, then $C_{sub} = \Gamma(B_{sub}) \cap C_i$ even though u is not adjacent to v in G'. Under this case, $\frac{w(C_{sub})}{w(B_{sub})} \geq \alpha_i$ by Lemma 1. It's a contradiction and $v \in V'_{j+1}$, which means $j < j_v$. ■

Chapter Four Other Approach

Up to now, all the possible cases have been considered so that my proof is complete.

However, the technique Dene Kernel Removal is such a powerful tool for analyzing the structure of a graph that some other approach for part of my conclusion is found. In this section, I deal with the problem from another point view and give the corresponding proof. Note that part of this proof is just the same as the one in Chapter Three, so that this chapter is just a supplement for presenting an original idea about analyzing the structure of a graph.

In Subsection 3.3, the situation that one delete an edge (u, v) in G is split into eleven cases according to the classification for the two vertices in G and G'. However, if we only consider the classification for the edge (u, v), the situation can be split into three subcases, which does not match in those eleven cases exactly. In precise, they are:

(1) $(u, v) \in B_i \times C_i$ with $\alpha_i < 1$ for some $i \in \{1,2, \dots, k\}$;

(2) $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$;

(3) $(u, v) \notin B_i \times C_i$ for $\forall i \in \{1,2, \dots, k\}$.

To make the problem clear, here I first show the relationship between these three cases and the analysis in Chapter Three. And for the convenience of discussion, the new three cases are called edge-cases, and the previous eleven cases are called vertex-cases in this chapter. It is easy to see that vertex-cases (1)-(3) are all belong to edge-case (2), and conversely, they are all possible subcases for edge-case (2). At the same time, vertex-cases (4)-(7) are all in edge-case (3). However, vertex-cases (8)-(11) may be in either edge-case (1) or edge-case (3). Meanwhile, vertex-cases (4)-(7) together with vertex-cases (8)-(11) are all subcases in edge-cases (2) and (3). As for my proof, it is obvious that the Theorem 1 in Chapter Three is exactly a proof for edge-case (3) so that there is no need to present such proof in this chapter. And the Lemma 2-5 are also needed for the following proof.

4.1 $(u, v) \in B_i \times C_i$ with $\alpha_i < 1$ for some $i \in \{1, 2, \ldots, k\}$

In this case, it is possible that after $(u, v) \in B_i \times C_i$ is deleted, the bottleneck

decomposition of G' becomes different (see, e.g. Figure 2 and 3). The main reason is that the neighborhood of u in C_i and the neighborhood of v in B_i will change.

Figure 2: One case showing that the bottleneck decomposition of the graph before and after deleting the edge (u, v) are different. It is easy to see that in the original graph G, u are in B_2 and v are in C_2, after deleting the edge $(u, v), u$ becomes a vertex of B_1' in G', and v becomes a vertex of B_3'.

Figure 3: One case showing that the bottleneck decomposition of the graph before and after deleting the edge (u, v) are different. It is easy to see that in the original graph G, u are in B_2 and v are in C_2, after deleting the edge $(u, v), u$ becomes a vertex of B_2' in G', and v becomes a vertex of B_1'.

To prove that u and v cannot improve their utilities by deleting edge $(u, v) \in B_i \times C_i, i = 1,2, \cdots, k$, the idea consist of two phases. First, we develop an understanding of feasible configurations, by Lemma 2 and Lemma 4. Next, we apply incentive analysis

to u and v to show neither of them can gain in utility. However, after the first step, the rest cases match the condition in Lemma 6 and Lemma 7 exactly, so it seems that we can omitted the precise proof. It worth mentioning that although the proof for the edge-case (1) is nearly the same as the one for vertex-cases, they are showing two different kind of thinking when dealing with the structure of a graph, based on the properties of vertices and of edges.

4.2 $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$

Since $\alpha_k = 1$, we know that $B_k = C_k$, and for $u, v \in B_k = C_k$, they have the same status. Thus the case $j_u < j_v$ is equivalent to $j_v < j_u$ and it is enough to discuss two cases that are $j_u < j_v$ and $j_u = j_v$.

4.2.1 Case $j_u < j_v$

Here we first show a lemma the properties of $u \in G'$.

Lemma 8 In the bottleneck decomposition of $G' = G - (u, v)$ where $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$, if $j_u < j_v$, then u must be a B'-class vertex

in G' and $\frac{w(\Gamma(B'_{j_u} \cap C_k) \cap B_k) - w(v)}{w(B'_{j_u} \cap C_k)} < 1$.

Proof. Since $j_u < j_v$, Lemma 2 guarantees that $V'_{j_u} = V_{j_u}$ and $\alpha'_{j_u} < 1$. Suppose to the contrary that u is in C'_{j_u}, then at least one of its neighbors, say x, must be in B'_{j_u}. Thus such a vertex x only can be from $C_i, i = j_u, \cdots, k$. Without loss of generality, define:

(1)$B'_{j_u1} = \cup_{l=j_u}^{k}(B'_{j_u} \cap C_l)$, $B'_{j_u2} = B'_{j_u} \setminus B'_{j_u1}$;

(2)$C'_{j_u1} = \cup_{l=j_u}^{k} \Gamma\left(B'_{j_u} \cap C_l\right) \cap B_l$, $C'_{j_u2} = C'_{j_u} \setminus C'_{j_u1}$.

Note that $j_u < j_v$, vertex v does not appear in (B'_{j_u}, C'_{j_u}). So for any vertex $x \in B'_{j_u} \cap C_k$, it could not be vertex v and u and its neighborhood in B_k are preserved in V'_{j_u} even though (u, v) is deleted. Thus B'_{j_u1} and C'_{j_u1} are well defined.

If we let $C = B'_{j_u} \cap C_l$ and $B = \Gamma\left(B'_{j_u} \cap C_l\right) \cap B_l$ for each $l = j_u, \cdots, k$, then $\frac{w(B)}{w(C)} \geq 1$. In addition, we have $\frac{w\left(C'_{j_u1}\right)}{w\left(B'_{j_u1}\right)} \geq 1 > \alpha'_{j_u}$ which implies that $\frac{w\left(C'_{j_u2}\right)}{w\left(B'_{j_u2}\right)} < \alpha'_{j_u}$. On the other hand, it is obvious that all vertices in B'_{j_u2} are from $B_i, i = 1, \cdots, k-1$ and Proposition 2-(1) promises that there is no edge between B'_{j_u2} and C'_{j_u1}. Thus the neighborhood $\Gamma\left(B'_{j_u2}\right) \cap V'_{j_u}$ is contained in C'_{j_u2} and we have $\frac{w\left(\Gamma\left(B'_{j_u2}\right) \cap V'_{j_u}\right)}{w\left(B'_{j_u2}\right)} \leq \frac{w\left(C'_{j_u2}\right)}{w\left(B'_{j_u2}\right)} < \alpha'_{j_u}$. It's a contradiction. Furthermore, we know there is no u's neighbor in B'_{j_u2} which leads to that u could not be in C'_{j_u}.

For the second statement, we know that $u \in B'_{j_u} \cap C_k$ and v is not in C'_{j_u}. Otherwise $\Gamma\left(B'_{j_u} \cap C_k\right) \cap B_k \subseteq C'_{j_u}$ and we can decrease α'_{j_u} by removing $B'_{j_u} \cap C_k$ and $\Gamma\left(B'_{j_u} \cap C_k\right) \cap B_k$ from B'_{j_u} and C'_{j_u}, respectively. So $\Gamma\left(B'_{j_u} \cap C_k\right) \cap B_k - \{v\} \subseteq C'_{j_u}$.

But if $\frac{w\left(\Gamma\left(B'_{j_u} \cap C_k\right) \cap B_k\right) - w(v)}{w\left(B'_{j_u} \cap C_k\right)} \geq 1 > \alpha'_{j_u}$, then we can get a new pair with a

smaller α-value by deleting all vertices in $B'_{j_u} \cap C_k$ from B'_{j_u} and all vertices in $\Gamma\left(B'_{j_u} \cap C_k\right) \cap B_k$ from C'_{j_u}, respectively. It's a contradiction.■

The result that $u \in B'_{j_u}$ and $\alpha'_u = \alpha'_{j_u} < 1$ in Lemma 8 leads to the following result directly.

Corollary 2. In the bottleneck decomposition of $G' = G - (u, v)$ where $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$, if $j_u < j_v$, then u's utility $U'_u < U_u$.

It is obvious that if v is still a B'-class vertex in G' when $j_u < j_v$, then its utility also cannot be improved. So next we shall focus on situation that v is a C'-class vertex if $j_u < j_v$ and the following claims are necessary.

Lemma 9. In the bottleneck decomposition of $G' = G - (u, v)$ where $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$, if $j_u < j_v$, then any $B'_j, j = j_u, \cdots, j_v$, does not contain vertices from $C_l, l = j_u, \cdots, k-1$.

Proof. By the proof in Lemma 8, it is easy to see that B'_{j_u} does not contain any vertex in $C_l, l = j_u, \cdots, k-1$ and this claim holds for $j = j_u$.

Suppose this result is correct for $j = j_u + 1, \cdots, j_v - 1$ and we consider the case $j = j_v$. Assume that there is one vertex $x \in B'_{j_v} \cap C_l \neq \emptyset, l = j_u, \cdots, k-1$, then $\Gamma(x) \cap B_l \subseteq V'_{j_v}$. Otherwise, at least one vertex $y \in \Gamma(x) \cap B_l$ is removed before step j_v. Since any $B'_j, j = j_u, \cdots, j_v - 1$, does not has the vertices from $C_l, l =, j_u, \cdots, k-1$, then y can not have neighbors in B'_j and it only can be a B'-class vertex. But once y is in some $B'_j, j < j_v$, then vertex x should be in C'_j and be deleted before step j_v which is

a contradiction.

Since $\Gamma(x) \cap B_l \subseteq V'_{j_v}$, then we can partition B'_{j_v} and C'_{j_v} as follows:

(1)$B'_{j_u 1} = \cup_{l=j_u}^{k-1}\left(B'_{j_u} \cap C_l\right), B'_{j_u 2} = B'_{j_u} \setminus B'_{j_u 1}$;

(2)$C'_{j_u 1} = \cup_{l=j_u}^{k-1} \Gamma\left(B'_{j_u} \cap C_l\right) \cap B_l,\ C'_{j_u 2} = C'_{j_u} \setminus C'_{j_u 1}$.

By using the same method in Lemma 8, we can get that $\frac{w\left(\Gamma\left(B'_{j_v 2}\right) \cap V'_{j_v}\right)}{w\left(B'_{j_v 2}\right)} \leq \frac{w\left(C'_{j_v 2}\right)}{w\left(B'_{j_v 2}\right)} < \alpha'_{j_v}$. It's a contradiction and this claim holds.■

The important role of Lemma 9 is that it indicates that it is possible that $B'_j \cap C_k \neq \emptyset, j = j_u, \cdots, j_v - 1$, and if v is in C'_{j_v}, its neighbors in B'_{j_v} only can be from $B_k = C_k$. The following result shall state that if v is a C'-class vertex, its α-value cannot be less than 1.

Lemma 10. In the bottleneck decomposition of $G' = G - (u, v)$ where $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$, if $j_u < j_v$ and v is in C'_{j_v}, then $\alpha'_{j_v} = 1$.

Proof. Suppose to the contrary that $\alpha'_{j_v} < 1$. For convenience, define disjoint sets:

(1) $A_{j_u} = \Gamma\left(B'_{j_u} \cap C_k\right) \cap B_k - \{v\}$;

(2)$A_{j_u+1} = \Gamma\left(B'_{j_u+1} \cap C_k\right) \cap B_k - A_{j_u}$;

⋮

(3)$A_{j_v} = \Gamma\left(B_{j_v} \cap C_k\right) \cap B_k - \cup_{j=j_u}^{j_v-1} A_j$.

Based on the assumption that $\alpha'_{j_v} < 1$, all sets $B'_j \cap C_k, j = j_u, \cdots, j_v$, are independent

and by the same method in Lemma 8, we have $w(A_j) < w(B'_j \cap C_k), j = j_u, \cdots, j_v$.

Therefore we get $\sum_{j=j_u}^{j_v} w(A_j) < \sum_{j=j_u}^{j_v} w(B'_j \cap C_k)$.

More importantly, if we let $C = \bigcup_{j=j_u}^{j_v}(B'_j \cap C_k) \subseteq B_k = C_k$, then $\bigcup_{j=j_u}^{j_v} A_j = \Gamma(C) \cap B_k$ since $v \in A_{j_v}$. But we know states that $w(\Gamma(C) \cap B_k) \geq w(C)$ which leads to a contradiction. So $\alpha'_{j_v} = 1$.■

Obviously, the following corollary can be obtained directly from Lemma 10.

Corollary 3. In the bottleneck decomposition of $G' = G - (u, v)$ where $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$, if $j_u < j_v$, then v's utility $U'_v \leq U_v$.

According to Corollary 2 and 3, we get our result for edge-case (2) that neither u nor v will not improve its utility by deleting the edge (u, v) under the condition of 4.2.1.

4.2.2 Case $j_u = j_v$

Similarly, we first present a lemma showing the classification of u and v in G'.

Lemma 11. In the bottleneck decomposition of $G' = G - (u, v)$ where $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$, we have:
(1) if $j_u = j_v = j < k$, then u and v are both B'-class vertices;
(2) if $j_u = j_v = j = k$, then u and v are both B'-class vertices or $u \in B'_k = C'_k = B_k = C_k, v \in B'_k = C'_k = B_k = C_k$.

Proof. (1) We still discuss four cases based on $u \in C_j'$ or $B_j', v \in C_j'$ or B_j'. It is worthy to note that the condition $j_u = j_v = j < k$ makes us conclude that $\alpha_j' \leq \alpha_j < \alpha_k = 1$.

Case 1: $u \in C_j'$ and $v \in C_j'$. Then B_j' is an independent set and it only contains vertices from $B_l, l = 1, \cdots, k-1$ by Lemma 4. But since u and v are in C_k, their neighbors must be in $C_l, l = 1, \cdots, k$, which means that there is no u and v's neighbors in B_j' and u and v cannot be in C_j'.

Case 2: $u \in B_j'$ and $v \in C_j'$. Since $v \in C_j'$, then $\Gamma(u) \cap B_k \subseteq C_j'$ even though (u, v) is deleted. In addition, $\Gamma(B_j' \cap C_k) \cap B_k \subseteq C_j'$. Thus we can partition B_j' and C_j' as follows:

(1)$B_{j1}' = \cup_{l=j_u}^{k}(B_j' \cap C_l)$; $B_{j2}' = B_j' \setminus B_{j1}'$;

(2)$C_{j1}' = \cup_{l=j_u}^{k}\Gamma(B_j' \cap C_l) \cap B_l$; $C_{j2}' = C_j' \setminus C_{j1}'$.

By applying the statement $\frac{w(\Gamma(B_j' \cap C_l) \cap B_l)}{w(B_j' \cap C_l)} \geq 1 > \alpha_j', l = j_u, \cdots, k$, and we can get a new pair $(B_{j2}', \Gamma(B_{j2}') \cap V_j')$ with a smaller α-value by removing all vertices in B_{j1}' from B_j' and all vertices in C_{j2}' from C_j'. It's a contradiction.

Case 3: $u \in C_j'$ and $v \in B_j'$. This case is equivalent to Case 2. So the proof is omitted.

Case 4: $u \in B_j'$ and $v \in B_j'$. For example in Figure 4, we can get $B_1 = \{v_1\}, C_1 = \{v_2\}$ with $\alpha_1 = \frac{3}{5}$ and $B_2 = C_2 = \{v_3, v_4, \cdots, v_7\}$ with $\alpha_2 = 1$. If $(u, v) = (v_4, v_3)$ is deleted in G', then $B_1' = \{v_1, v_3, v_4\}$ and $C_1' = \{v_2, v_5, v_6, v_7\}$ with $\alpha_1' = \frac{3}{5}$.

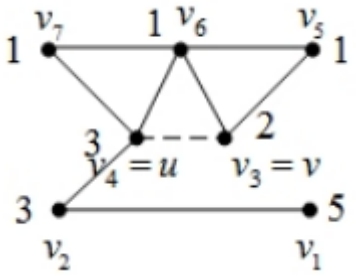

Figure 4: One case of deleting an edge $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$ from the original graph G to get G' in which u and v both belong to B_j'.

(2) If $j = k$, then it is impossible that $u \in B_k'$ and $v \in C_k'$ when $\alpha_k' < 1$ by the same method for Case 2 above. But if $\alpha_k' = 1$, then it is easy to get that $B_k' = C_k' = B_k = C_k$ since $V_k' = V_k$.

If $u \in C_k'$ and $v \in C_k'$ with $\alpha_k' < 1$, then there is no C-class vertex in B_k' by Lemma 4. But the fact that $V_k' = V_k = B_k = C_k$ and all vertices in $B_k = C_k$ are viewed as C-class vertex makes $B_k' = \emptyset$. So $\alpha_k' = 1$ and under this case, we have $B_k' = C_k' = B_k = C_k$. ■

Note that Figure 5 shows that it is possible that $u \in B_k'$ and $v \in B_k'$ with $\alpha_k' < 1$ in which $B_1 = C_1 = \{v_1, \cdots, v_5\}$ with $\alpha_1 = 1$ in G while $B_1' = \{v_1, v_2\}, C_1' = \{v_3, v_4, v_5\}$ with $\alpha_1' = \frac{3}{5}$ in G'.

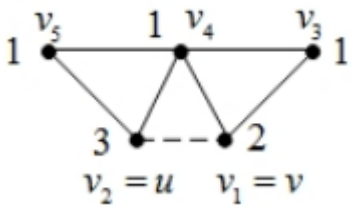

Figure 5: One case in which for the bottleneck decomposition of the graph G, we have $\alpha_k = 1$, but for the bottleneck decomposition of the graph G' who is the same as G but an edge (u, v), we have $\alpha_{k'}, < 1$.

New it is easy to get our result for condition 4.2.2.

Corollary 4. In the bottleneck decomposition of $G' = G - (u, v)$ where $(u, v) \in B_k \times C_k$ with $\alpha_k = 1$, if $j_v = j_u$, then the utilities of u and v cannot be improved.

Proof. For the case that $j_u = j_v = j \leq k$. If u and v are both B'-class vertices, then $\alpha'_u = \alpha'_j \leq 1$ and $\alpha'_v = \alpha'_j \leq 1$. So $U'_u = \alpha'_u \cdot w(u) \leq U_u$ and $U'_v = \alpha'_v \cdot w(v) \leq w(v) = U_v$. For the case that $u \in B_k = C_k$ and $v \in B_k = C_k, U'_u = U_u$ and $U'_v = U_v$. Thus this claim holds.■

Chapter Five Incentive Ratio of Bandwidth Sharing

Chapter Three and Four are talking about the truthfulness of bandwidth sharing of P2P under the condition that the only strategic play is through misreporting a connection to neighborhood. Although this is the most common way the cheat in a P2P network, it is possible that agent in the network use other method for possible utility gaining. Actually, such kind of cheating way has been found. In this Chapter, I will show one of such method with an example. And to make things clear, I will also give some basic proposition for that. However, the properties of the P2P network with proportional response mechanism will be my future work.

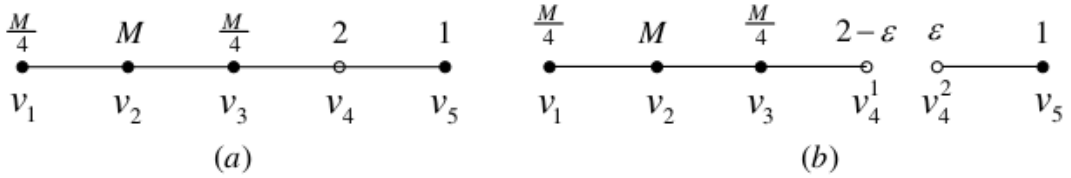

Figure 6: An example that player v_4 can improve its benefit by manipulation.

As shown in Figure 6-(a), there is a path G containing five vertices $v_1, v_2, \dots, v5$. The weights of all vertices

are $w(v_1) = w(v_3) = \frac{M}{4}, w(v_2) = M, w(v_4) = 2$ and $w(v_5) = 1$, where M is a large enough number. The bottleneck decomposition of G is $\mathcal{B} = \{(B_1, C_1)\}$, and in precise, $B_1 = \{v_2, v_4\}$and $C_1 = \{v_1, v3, v_5\}$. It is easy to check that $lim_{M\to\infty}\alpha_1 = \frac{1}{2}$, and the utility of v_4 tens to 1 as M tends to the infinity. Suppose vertex v_4 strategically allocates $2-\epsilon$ to v_3 and ϵ to v_5 respectively, where ϵ is an arbitrarily small positive number. Under this case, we construct a new graph G' shown in Figure 6-(b) and its bottleneck decomposition is $\mathcal{B}' = \{(B_1', C_1'), (B_2', C_2')\}$. In precise, $B_1' = \{v_5\}, C_1' = \{v_4^2\}$ with $\alpha_1' = \epsilon$ and $B_2' = \{v_2, v_4^1\}, C_2' = \{v_1, v_3\}$ with α_2' tends to$\frac{1}{2}$ as M tends the infinity. So $lim_{M\to\infty}U'_{v_4^1} = 1$ and $lim_{M\to\infty}U'_{v_4^2} = 1$ which means that v_4 obtains a larger utility 2 by such a strategic play.

We call this kind of strategic play *manipulation*. It can be modeled as following: One player v may manipulate its bandwidth allocation by fixing the bandwidth amount to each neighbor which may not be proportional to what it receives in advance. To be specific, assume $\Gamma(v) = \{u_1, u_2, \dots, u_{d_v}\}$ and player v distributes its bandwidth as $w_v^i = x_{vu_i}, i = 1,2, \dots, d_v,$ with $\sum_{i=1}^{d_v} w_v^i = w_v$. Other players still follow the proportion response protocol to upload their bandwidth.

It is obvious that the bandwidth allocation mechanism changes completely because of the manipulation. In order to analyze player v's utility after the manipulation, we

should obtain a new allocation X' which comes from a market equilibrium in advance. Specially, in such a new allocation X', player v's distribution is fixed and other players follow the proportional response protocol. The main idea is to compute the market equilibrium from the bottleneck decomposition of an altered network G' such that this new market equilibrium's allocation satisfies the distribution requirement of each player. Now we construct a new network G' as follows. First, vertex v is duplicated into d_v copies $v_1, v_2, \dots, v_{d_v}$ in G' and the weight of copy v_i is denoted as $w_v^i, i = 1,2, \dots, d_v$. Next if $(v, u_i) \in E(G)$, then an edge between v_i and $u_i, i = 1,2, \dots, d_v$ is built in G'. Therefore $V(G') = (V(G) - v) \cup (\bigcup_{i=1}^{d_v} v^i)$ and $E(G') = \left(E(G) - \bigcup_{i=1}^{d_v}(v, u_i)\right) \cup (\bigcup_{i=1}^{d_v}(v^i, u_i))$. Obviously each player v_i is a leaf in G'. Notice that the new utility of player v in X' is $U'_v = \sum_{i=1}^{d_v} U'_{v^i}$.

At the same time, when analyzing the properties of the bandwidth sharing when the mechanism is not truthful, a concept named *incentive ratio* is usually the most important. Generally speaking, the incentive ratio of the P2P network we consider can be defined as the maximal ratio ξ_v among all the vertex $v \in V$ satisfying $\xi_v = \max_{possible\ allocation}\{\frac{U'_v}{U_v}\}$, i.e. $\xi = \max_{v \in V} \xi_v$. A conjecture for the incentive ratio of P2P network bandwidth sharing under manipulation is 2, and this may showed in some future work.

Chapter Six Conclusion

In this article, we discuss the issue of possible cheating strategies of agent with respect to the proportional response mechanism for the application of bandwidth sharing.

We show that, if every agent follow this strategy, no player could gain by removing a connection arc from its neighbors. In other words, no player could gain eventually, if it removes one or more of its edge from the network environment.

The result, building on discrete mathematical techniques, resolves a long time unsolved problem whether the proportional response protocol is truthful. It is the first, to the best of our knowledge, non-trivial result on truthful mechanism for a practical network bandwidth resource sharing scheme.

REFERENCE

[1] Adsul B, Babu C, Garg J, et, al. Nash Equilibria in Fisher Market[C]//Algorithmic Game Theory. Springer, 2010: 30-41.

[2] Br ânzei S, Chen Y, Deng X, et al. The Fisher market game: Equilibrium and welfare[C]//Twenty-Eighth Conference on Artificial Intelligence. 2014: 587-593.

[3] Condon T, Dahl H, Devarajan S. IMPLEM&~ TING A COMPUTABLE GENERAL EQUILIBRIUM MODEL ON GAMS THE CAUEROON MODEL[J]. 1986.

[4] Chen N, Deng X, Zhang J. How profitable are strategic behaviors in a market?[M]//Algorithms–ESA 2011. Springer Berlin Heidelberg, 2011: 106-118.

[5] Chen N, Deng X, Zhang H, et al. Incentive ratios of fisher markets[M]//Automata, Languages, and Programming. Springer Berlin Heidelberg, 2012: 464-475.

[6] Deng X, Papadimitriou C, Safra S. On the complexity of equilibria[C]//Proceedings of the thiry-fourth annual ACM symposium on Theory of computing. ACM, 2002: 67-71.

[7] Devanur N R, Papadimitriou C H, Saberi A, et al. Market equilibrium via a

primal-dual-type algorithm[C]//2013 IEEE 54th Annual Symposium on Foundations of Computer Science. IEEE Computer Society, 2002: 389-389.

[8] Hurwicz L. On informationally decentralized systems[J]. Decision and organization, 1972.

[9] Iosifidis G, Gao L, Huang J, et al. Enabling crowd-sourced mobile Internet access[C]//INFOCOM, 2014 Proceedings IEEE. IEEE, 2014: 451-459.

[10] Iosifidis G, Gao L, Huang J, et al. Incentive mechanisms for user-provided networks[J]. Communications Magazine, IEEE, 2014, 52(9): 20-27.

[11] Jackson M O, Manelli A M. Approximately competitive equilibria in large finite economies[J]. Journal of Economic Theory, 1997, 77(2): 354-376.

[12] Jain K. A polynomial time algorithm for computing an Arrow-Debreu market equilibrium for linear utilities[J]. SIAM Journal on Computing, 2007, 37(1): 303-318.

[13] Lange O. The computer and the market[J]. 1967, 1967: 158-61.

[14] Otani Y, Sicilian J. Equilibrium allocations of Walrasian preference games[J]. Journal of Economic Theory, 1982, 27(1): 47-68.

[15] Papadimitriou C. Algorithms, games, and the internet[C]//Proceedings of the thirty-third annual ACM symposium on Theory of computing. ACM, 2001: 749-753.

[16] Roberts D J, Postlewaite A. The incentives for price-taking behavior in large exchange economies[J]. Econometrica: Journal of the Econometric Society, 1976: 115-127.

[17] Roth A E, Sönmez T, Ünver M U. Pairwise kidney exchange[J]. Journal of Economic theory, 2005, 125(2): 151-188.

[18] Schollmeier R. A definition of peer-to-peer networking for the classification of peer-to-peer architectures and applications[C]//Peer-to-Peer Computing, 2001. Proceedings. First International Conference on. 2001: 101-102.

[19] Teng F, Magoules F. Resource pricing and equilibrium allocation policy in cloud computing[C]//Computer and Information Technology (CIT), 2010 IEEE 10th

International Conference on. IEEE, 2010: 195-202.

[20] Wu F, Zhang L. Proportional response dynamics leads to market equilibrium[C]//Proceedings of the thirty-ninth annual ACM symposium on Theory of computing. ACM, 2007: 354-363.

[21] Ye Y. A path to the Arrow–Debreu competitive market equilibrium[J]. Mathematical Programming, 2008, 111(1-2): 315-348.

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