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论文题目： Estimation of shock reflection by large-angle wedges for self-similar potential flow near sonic arc

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Estimation of shock reflection by large-angle wedges for self-similar potential flow near sonic arc

ABSTRACT

Shock reflection problem is a fundamental and important issue both in theoretical analysis and in applications. It is also a building block for general study of multi-dimensional conservation laws. In this thesis we start with surveying various shock reflection-diffraction patterns, then formulate the regular reflection of shock by wedge for potential flow as a free boundary nonlinear problem of mixed-composite hyperbolic-elliptic type. Our task in this thesis is to give a direct and simple proof for the general estimate and the regularity results for the global solution obtained by Chen and Feldman in [9][10]. More precisely, it is $C^{1,1}$-regularity at the point where the pseudo-sonic circle meets the reflected shock and belongs to the $C^{2,\alpha}$-regularity up to the pseudo-sonic circle in the pseudo-subsonic region. The main idea in the proof is to develop the technique of Maximum principle to handle free boundary problems. At the end of the thesis, we would also address several open questions in general shock reflection problem.

Keywords: Shock regular reflection, Potential flow, free boundary problem, regularity

Contents

Chapter 1 Introduction

1.1 Description of shock reflection

Shock waves may occur in many physical or natural situations, for example, solar winds of sun can generate bow shocks(due to a planet's magnetosphere interacting with a stellar wind[15][16]), see Fig 1.1 (cited from Wikipedia), aircraft can generate supersonic or almost sonic shocks[17], see Fig 1.2 (cited from Wikipedia) and various explosions also generate blast waves[18], see Fig 1.3 (cite from Wikipedia). When a shock hits different types obstacles, shock reflection-diffraction phenomena may occur. Moreover, shock reflection-diffraction by a flat-boundary is one of the most fundamental multi-dimensional problems in mathematical fluid dynamics. When a plane shock hits the wedge and heads on, a self-similar shock of reflection-diffraction moves outward as the original shock waves forward in time. The solutions of this problem is showed to be fundamental for the mathematical theory of multidimensional hyperbolic systems of conservation laws which is still largely incompletely.

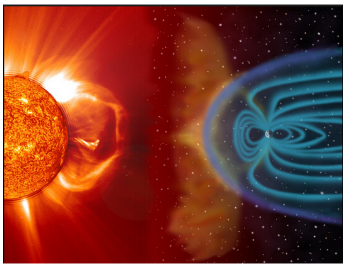

Figure 1.1 bow shock

Figure 1.2 aircraft

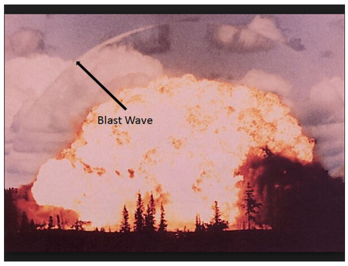

Figure 1.3 blast wave

The complexity of reflection-diffraction configurations is first proposed by Ernst Mach[1] in nineteenth century, in which he addressed two kinds of pattern may occur if shock waves hits a wedge: regular reflection (simpler figuration, occur when wedge angle is relatively large) and Mach reflection (complicated figuration). Then due to the importance in wide application, the problem attracted many scientists' attention. However, it was unlucky that this problem remained dormant until mid twentyth century when von Neumann, Friedrichs, Bathe, and many experimental, computational, and asymptotic analysts began extensive and deep research into almost every aspects of shock reflection-diffraction phenomena. See von Neumann[2][3] and Ben-Dor[4]; also see [19][20][21][22]. A few decades ago, it had been showed the complete situation is much more complicated than Mach originally observed, for example, the Mach reflection can be further divided into more sub-patterns (such as, irregular Mach reflection), and various other patterns of shock reflection-diffraction may also be produced like the von Neumann reflection and the Guderley reflection see[5][6][7][8]. Lately in 2005, Chen and Feldman [9][10] obtained the first global existence theory of shock reflection configurations for potential flow with the wedge angle θ_w is large (close to 90 degree). For stability, they also proved that constructed solution converges to the unique solution of the normal reflection when θ_w tends to $\frac{\pi}{2}$. In 2009, Bae, Chen and Feldman[11] showed the regularity of solutions to regular shock reflection for potential flow and argued the transition of the different patterns of shock reflection configurations.

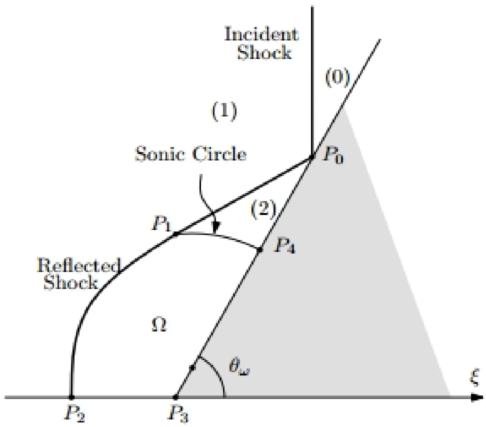

Figure 1.4 Regular Reflection-Diffraction Configuration

But there are still many essential issues need to be studied before fully understanding the phenomena of shock reflection, which includes the following two problems.

(1) The dependence of the reflection patterns upon some physical parameters, for example the wedge angle θ_w. Various criteria and conjectures have been proposed in determining the existence of configurations for the patterns, for examples,

(1.1) **von Neumann's detachment conjecture[23]:** Regular reflection configuration may exist globally whenever the two-shock configuration exists locally around the incident point P_0 (see Fig 1.4, cite from[10]). More specifically, there is a detachment angle θ_d so that when $\theta_w \in (\theta_d, \frac{\pi}{2})$ the global regular reflection exists and it will transit to Mach reflection when $\theta_w \in (0, \theta_d)$.

(1.2) **von Neumann's sonic conjecture[23]:** For $\theta_w \in (\theta_d, \frac{\pi}{2})$, the regular reflection configuration may still not be stable. But there is a sonic angle θ_{sonic} satisfying $0 < \theta_d < \theta_{sonic} < \frac{\pi}{2}$ such that a stable regular reflection exists when $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$.

We can note that in fact state (2) behind the shock only exists when $\theta_w \in (\theta_d, \frac{\pi}{2})$ (this is the motivation of detachment conjecture). Moreover, for each $\theta_w \in (\theta_d, \frac{\pi}{2})$ there exists two possible state (2) (weak sense and strong sense, with $\rho_2^{weak} < \rho_2^{strong}$). We always choose weak state (2) since Elling (2011) proved that for strong state (2) the global regular reflection solution fails to exist. Besides, the motivation of sonic conjecture is based on the following fact: there will exist $\theta_{sonic} \in (\theta_d, \frac{\pi}{2})$ such that: state (2) is supersonic at P_0 for $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$, while state (2) is subsonic at P_0 for $\theta_w \in (\theta_d, \theta_{sonic})$. If it is supersonic, the propagation speeds are finite and the state (2) is completely determined by the local information: state (1), state (0), and the location of the point P_0. Fairly to say, the disturbance information at the corner point P_3 (see Fig 1.4) cannot travel towards the incident point P_0. However, if it is subsonic the disturbance information can reach P_0 and potentially altering the reflection-diffraction type. For simplicity, we study the case that the regular reflection configuration is stable and converges to the uniquely determined normal reflection while $\theta_w \to \frac{\pi}{2}$, i.e., the reflection point P_0 is supersonic.

It is clear to see the sonic conjecture is stronger than the detachment one. However, Sheng-Yin [12] point out the regime between the angle θ_d and θ_{sonic} is very narrow and can be neglected. For simplicity we assume $\theta_w \in (\theta_{sonic}, \frac{\pi}{2})$ in the following of this thesis.

(2) Transition criteria between different patterns of shock reflection-diffraction configurations.

One should note that physical and numerical experiments are hampered here since the dissipation or physical viscosity smear the shock, meanwhile the boundary layers interact with the reflection patterns and cause spurious Mach steams[1]. With noticing this fact, it seems that an ideal way to analyse the full shock reflection patterns, especially the transition criteria, is still by a rigorous mathematical analysis.

1.2 Overview of my work

In this thesis, we first state the fundamental issues behind the shock reflection phenomena following with rigorously formulated model. Especially, we explain the ideas behind different pattern-transition criteria. Besides, we point out the similarity and difference between potential flow and Euler flow for estimation near sonic arc.

Second we systematically itemize the results obtained by Chen, Feldman[9][10]. Suppose a stable global solution exists, we specifically describe boundary conditions for the formulated free boundary problem in three different coordinates. Moreover, we point out the idea of introducing ellipticity cutoff with different choice of constant.

The distinguished work we manage to do is to give a simple and direct proof for $\varphi \geq \varphi_2$, $\psi \leq Cx^2$, $\psi \leq \frac{2}{3(\gamma+1)}x^2$, $C^{1,1}$-regularity up to sonic arc and $C^{2,\alpha}$-regularity up to the sonic arc away from shock point P_1 by developed Maximum principle and scaling techniques.

1.3 Outline of this thesis

In Chapter 2, we first formulate the shock reflection problem as a nonlinear initial-boundary value problem by introducing potential flow and then adopt self-similarity of solution to reformulate the problem into a boundary value problem in the unbounded domain. Noting the boundary conditions of the subsonic domain are overdetermined[9] for the elliptic equation since the conditions on the rest boundaries are prescribed except the sonic arc, we thus transform the problem to free boundary problem and employ the free boundary techniques. In chapter 3, we present the unique solution of normal reflection when the wedge angle is $\pi/2$ and describe a global theory for regular reflection structure for potential flow obtained by Chen, Feldman [9][10] and Bea, Chen, Feldman[11]. In Chapter 4, we concentrate on the estimation near the sonic arc, specifically we will give a new but simple proof by Maximum principle for some classical results obtained in [9][10]. We will primarily talk and give direct proofs about the higher regularity near sonic arc away from shock in Chapter 5. At the end of this thesis, we conclude this paper and point out some works that remains to be open.

Chapter 2 Mathematical formulation of shock reflection problem

In this chapter, I formulate the shock reflection problem by wedge when the wedge angle is suitably large. The full Euler equation for compressible fluids in $R^3_+ := R_+ \times R^2$, where $t \in R_+$ and $\mathbf{x} \in R^2$, consist of the conservation law of mass, momentum and energy of the form:

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0, \\ \partial_t (\rho \mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0, \\ \partial_t (\frac{1}{2}\rho |\mathbf{v}|^2 + \rho e) + \nabla_{\mathbf{x}} \cdot ((\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e + p)\mathbf{v}) = 0, \end{cases} \tag{2-1}$$

where $\mathbf{v} = (u, v)$ is the fluid velocity, ρ the density, p the pressure, and e the internal energy. Note there are other two important thermodynamic variables, temperature θ and energy S. Choosing (ρ, S) as the independent variables, then the constitutive connection can be written as $(p, \theta, e) = (p(\rho, S), \theta(\rho, S), e(\rho, S))$ governed by the relation

$$\theta dS = de - \frac{p}{\rho^2} d\rho.$$

When a flow is potential, that is, there is a velocity potential Φ such that

$$\mathbf{v} = \nabla_{\mathbf{x}} \Phi$$

Then the Euler equations for the flow consist of conservation law of mass and the Bernoulli law for the density ρ and the velocity potential Φ:

$$\begin{cases} \partial_t \rho + div_X (\rho \nabla_{\mathbf{x}} \Phi) = 0, & \text{(conservation of mass)} \\ \partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}} \Phi|^2 + i(\rho) = B_0, & \text{(Bernoulli's law)} \end{cases} \tag{2-2}$$

where B_0 is the Bernoulli constant determined by the incoming flow and(or) boundary conditions, and

$$i'(\rho) = \frac{p'(\rho)}{\rho} = \frac{c^2(\rho)}{\rho}$$

with $c(\rho)$ being the sound speed. For polytropic gas, $p(\rho) = \kappa\rho^\gamma, c^2(\rho) = \kappa\gamma\rho^{\gamma-1}, \gamma > 1$. Without loss of generality, we choose $\kappa = \frac{\gamma-1}{\gamma}$ such that

$$i(\rho) = \rho^{\gamma-1}, \qquad c^2(\rho) = (\gamma-1)\rho^{\gamma-1} \tag{2-3}$$

which can be achived by the scaling:$(\mathbf{x}, t, B_0) \to (\alpha\mathbf{x}, \alpha^2 t, \alpha^{-2}B_0), \alpha^2 = \frac{\kappa\gamma}{\gamma-1}$.

Besides, the second equation in (2-2) can be written as

$$\rho(D\Phi) = i^{-1}(B_0 - (\partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2)).$$

For a steady solution $\Phi = \varphi(\mathbf{x})$, i.e., $\partial_t\Phi = 0$, we can further obtain the famous steady potential flow equation in aerodynamics:

$$\nabla_{\mathbf{x}} \cdot (\rho(\nabla_{\mathbf{x}}\Phi)\nabla_{\mathbf{x}}\Phi) = 0 \tag{2-4}$$

2.1 Initial-boundary value problem

When a plane shock in the ($\mathbf{x}$,t)-coordinates, $\mathbf{x} = (x_1, x_2) \in \mathbf{R}^2$, with left state $(\rho, \nabla_{\mathbf{x}}\Phi) = (\rho_1, u_1, 0)$ and right state $(\rho_0, 0, 0), u_1 > 0, \rho_0 < \rho_1$, hits a symmetric wedge

$$W := \{(x_1, x_2) : |x_2| < x_1 \tan\theta_w, x_1 > 0\}$$

head on, it experiences a reflection-diffraction process, where $\theta_w \in (0, \frac{\pi}{2})$ is the wedge half-angle. Then the Bernoulli law (2) becomes

$$\partial_t\Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2 + i(\rho) = i(\rho_0) \tag{2-5}$$

This reflection-diffraction problem can be formulated as the following mathematical problem.

Problem 2.1 (Initial-Boundary Value Problem)**.** *Seek a solution of the system of equations (1) and (2), the initial condition at $t = 0$:*

$$(\rho, \Phi)|_{t=0} = \begin{cases} (\rho_0, 0) & \textit{for } |x_2| > x_1 \tan\theta_w, x_1 > 0 \\ (\rho_1, u_1 x_1) & \textit{for } x_1 < 0 \end{cases} \tag{2-6}$$

and the slip boundary condition along the wedge boundary ∂W:

$$\nabla\Phi \cdot \upsilon|_{\partial W} = 0 \tag{2-7}$$

where υ is the exterior unit normal to ∂W (see Fig 2.1).

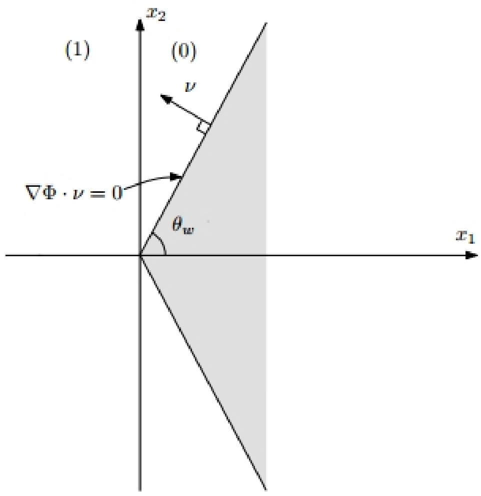

Figure 2.1 Initial-Boundary Value Problem

2.2 Boundary value problem

Notice that Problem 2.1 is invariant under the self-similar scaling:

$$(\mathbf{x}, t) \to (\alpha\mathbf{x}, \alpha t), \quad (\rho, \Phi) \to (\rho, \Phi/\alpha) \qquad \text{for} \quad \alpha > 0$$

Thus, we seek a self-similar solution with the form:

$$\rho(\mathbf{x}, t) = \rho(\xi, \eta), \quad \Phi(\mathbf{x}, t) = t\phi(\xi, \eta), \quad \text{for} \quad (\xi, \eta) = \mathbf{x}/t.$$

Let $(U, V) = (u - \xi, v - \eta)$ to be the pseudo-velocity, and $q = \sqrt{U^2 + V^2}$. Then the self-similar solutions are governed by the following system:

$$\begin{cases} (\rho U)_\xi + (\rho V)_\eta + 2\rho = 0, \\ (\rho U^2 + p)_\xi + (\rho UV)_\eta + 3\rho U = 0, \\ (\rho UV)_\xi + (\rho V^2 + p)_\eta + 3\rho V = 0, \\ (U(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma-1}))_\xi + (V(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma-1}))_\eta + 2(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma-1}) = 0. \end{cases} \tag{2-8}$$

We can calculate the eigenvalues of system (2-8):

$$\lambda_1 = \lambda_2 = \frac{U}{V}, \quad \lambda_\pm = \frac{UV \pm c\sqrt{q^2 - c^2}}{U^2 - c^2},$$

with the sonic speed $c = \sqrt{\frac{\gamma p}{\rho}}$.

Remark 2.2. *when the potential flow is pseudo-subsonic, i.e., $q < c$, the eigenvalues $\lambda_\pm$ are thus complex and the system consists of two nonlinear equations of hyperbolic-elliptic mixed type and two transport equations. Therefore, system (2-8) is hyperbolic-elliptic composite mixed in general.*

For the pseudo-potential function $\varphi = \phi - \frac{1}{2}(\xi^2 + \eta^2)$, we can calculate it is governed by the following potential flow equation of second order:

$$div(\rho(|D\varphi|^2, \varphi)D\varphi) + 2\rho(|D\varphi|^2, \varphi) = 0 \tag{2-9}$$

with

$$\rho(|D\varphi|^2, \varphi) = (\rho_0^{\gamma-1} - (\gamma - 1)(\varphi + \frac{1}{2}|D\varphi|^2))^{\frac{1}{\gamma-1}} \tag{2-10}$$

where the divergence div and gradient D are with respect to the self-similar variables (ξ, η). Then we have

$$c^2 = c^2(|D\varphi|^2, \varphi, \rho_0^{\gamma-1}) = \rho_0^{\gamma-1} - (\gamma - 1)(\frac{1}{2}|D\varphi|^2 + \varphi) \tag{2-11}$$

Therefore the equation (2-9) is a nonlinear equation of mixed elliptic-hyperbolic equation. By Remark 2.2, it is elliptic if and only if

$$|D\varphi| < c(|D\varphi|^2, \varphi, \rho_0^{\gamma-1}), \tag{2-12}$$

which is equivalent to

$$|D\varphi| < c_*(\varphi, \rho_0, \gamma) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma-1)\varphi)} \tag{2-13}$$

Shocks are discontinuities in the pseudo-velocity $D\varphi$. That is, if Ω^+ and $\Omega^- := \Omega \backslash \overline{\Omega^+}$ are two nonempty open subsets of $\Omega \subset R^2$ and $S := \partial\Omega^+ \cap \Omega$ is a C^1-curve where $D\varphi$ has a jump, then $\varphi \in W^{1,1}_{loc} \cap C^1(\Omega^\pm \cup S) \cap C^2(\Omega^\pm)$ is a global weak solution of (2-9) in Ω if and only if φ is in $W^{1,\infty}_{loc}(\Omega)$ and satisfies equation (2-9) in $\Omega^\pm$ and the continuity and Rankine-Hugoniot condition on S:

$$\begin{cases} [\varphi]_S = 0 & \text{continuity condition,} \\ [\rho(|D\varphi|^2, \varphi) D\varphi \cdot \upsilon]_S = 0 & \text{Rankine-Hugoniot jump condition.} \end{cases} \tag{2-14}$$

Here the continuity of φ is followed by the continuity of the tangential derivative of φ across S, which is a direct corollary of irrotionality of the velocity. The discontinuity S of $\nabla\varphi$ is called a shock if φ further satisfies the physical entropy condition that the corresponding density function $\rho(|\nabla\varphi|^2, \varphi, \rho_0)$ increases across S in the quasiflow direction. Chen-Feldman [9] remarked that the Rankine-Hugoniot jump condition with the continuity condition (2-14) across a shock for (2-9) is also fairly good approximation to the corresponding Rankine-Hugoniot conditions even for the full Euler equations for the shock with small strength since the errors are third-order in strength of the shock.

Fix constants $\rho_1 > \rho_0 > 0$. The plane incident shock solution in the $(\mathbf{x}, t)$-coordinates with states $(\rho, \nabla_{\mathbf{x}}\Phi) = (\rho_0, 0, 0)$ and $(\rho_1, u_1, 0)$ corresponds to a continuous weak solution φ of (2-9) in the self-similar coordinates (ξ, η) with the following form:

$$\varphi_0(\xi, \eta) = -\frac{1}{2}(\xi^2 + \eta^2) \quad for \quad \xi > \xi_0 \tag{2-15}$$

$$\varphi_1(\xi,\eta) = -\frac{1}{2}(\xi^2+\eta^2) + u_1(\xi-\xi_0) \quad for \quad \xi > \xi_0 \tag{2-16}$$

respectively, where

$$u_1 = (\rho_1-\rho_0)\sqrt{\frac{2(\rho_1^{\gamma-1}-\rho_0^{\gamma-1})}{(\gamma-1)(\rho_1^2-\rho_0^2)}} > 0 \tag{2-17}$$

$$\xi_0 = \rho_1\sqrt{\frac{2(\rho_1^{\gamma-1}-\rho_0^{\gamma-1})}{(\gamma-1)(\rho_1^2-\rho_0^2)}} = \frac{\rho_1 u_1}{\rho_1-\rho_0} > 0 \tag{2-18}$$

are the velocity of state (1) and the location of the incident shock, uniquely determined by (ρ_0,ρ_1,γ) with Rankine-Hugoniot jump condition along S, i.e. (2-14). Then $P_0 = (\xi_0, \xi_0\tan\theta_\omega)$ in the following Fig 2.2. Since the problem is symmetric with respect to the axis $\eta = 0$, it suffices to consider the problem in the half plane $\eta > 0$ outside the half-wedge

$$\Lambda := \{\xi < 0, \eta > 0\} \cup \{\eta > \xi\tan\theta_\omega, \xi > 0\}$$

Then the initial-boundary value problem in the $(\mathbf{x},t)$-coordinates can be formulated as the following boundary value problem in the self-similar coordinates (ξ,η).

Problem 2.3 (Boundary Value Problem)**.** *(see Fig 2.2) Seek a solution φ of equation (7) in the self-similar domain Λ with the slip boundary condition on $\partial\Lambda$:*

$$D\varphi\cdot v|_{\partial\Lambda} = 0 \tag{2-19}$$

and the asymptotic boundary condition at infinity:

$$\varphi \to \bar{\varphi} := \begin{cases} \varphi_0 & for\ \xi > \xi_0, \eta > \xi\tan\theta_\omega \\ \varphi_1 & for\ \xi < \xi_0, \eta > 0 \end{cases} \qquad when \quad \xi^2+\eta^2 \to \infty \tag{2-20}$$

where (2-20) holds in the sense that $\lim_{R\to\infty} ||\varphi - \overline{\varphi}||_{C(\Lambda\setminus B_R(0))} = 0$

It is convinced that the solutions of Problem 2.3 contain all possible patterns of shock reflection-diffraction configurations as observed in numerical and even physical

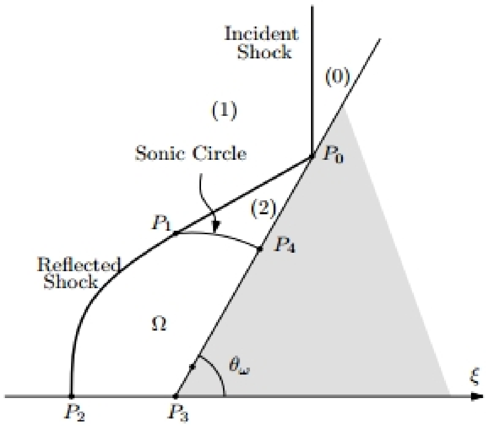

Figure 2.2 Regular Reflection

experiments. See[4][5][6][14].

2.3 Free boundary problem

Since φ_1 does not satisfy the slip boundary condition (2-19), the solution must differ from φ_1 in $\{\xi < \xi_0\} \cap \Lambda$ and thus a shock diffraction by the wedge occurs. In Chen-Feldman[9][10] the existence of global solution φ with its stability to Problem 2.3 has been established for potential flow when the wedge angle θ_ω is large and close to $\pi/2$, and the corresponding structure of solution is as follows (see Fig 2.2): The vertical line is the incident shock $S = \xi = \xi_0$ that hits the wedge at the point $P_0 = (\xi_0, \xi_0 \tan\theta_\omega)$, and state (0) and state (1) ahead of and behind S are given by φ_0 and φ_1 defined in (2-15) and (2-16), respectively. The solution φ and φ_1 differ within $\{\xi < \xi_0\}$ only in the domain $P_0P_1P_2P_3$ because of shock diffraction by the wedge vertex, where the curve $P_0P_1P_2$ is the reflected shock with the straight segment P_0P_1. State (2) behind P_0P_1 is of the form:

$$\varphi_2(\xi,\eta) = -\frac{1}{2}(\xi^2+\eta^2) + u_2(\xi-\xi_0) + (\eta - \xi_0\tan\theta_\omega)u_2\tan\theta_w \tag{2-21}$$

which satisfies

$$D\varphi \cdot \upsilon = 0 \quad on \quad \partial\Lambda \cap \{\xi > 0\};$$

the constant velocity u_2 and the angle between P_0P_1 and the ξ-axis are determined by $(\theta_\omega, \rho_0, \rho_1, \gamma)$ from the Rankine-Hugoniot jump condition and the continuity condition (2-14) for φ_1 and φ_2 across P_0P_1. Moreover, the constant ρ_2 of state (2) satisfies $\rho_2 > \rho_1$, and state (2) is pseudo-supersonic at the point P_0. In addition, $u_2 > 0$ when $\theta_\omega < \frac{\pi}{2}$. The solution φ is pseudo-subsonic within the pseudo-sonic circle for state (2) with the center $(u_2, u_2 \tan\theta_\omega)$ and radius $c_2 = \rho_2^{(\gamma-1)/2} > 0$ (the sonic speed of state (2)), and φ is pseudo-supersonic outside this circle containing the arc P_0P_1 in Fig 2.2, so that φ_2 is the unique solution in the domain $P_0P_1P_4$, which is the result as I discussed in chapter 1.1. Then φ differs from φ_2 in the domain $\Omega = P_1P_2P_3P_4$, where the equation is elliptic. Also we introduce the following notation for various parts of $\partial\Omega$:

$$\Gamma_{sonic} := \partial\Omega \cap \partial B_{c_2}(u_2, u_2\tan\theta_\omega) \equiv P_1P_4;$$
$$\Gamma_{shock} := P_1P_2;$$
$$\Gamma_{symm} := \{\eta = 0\} \cap \partial\Omega \equiv P_2P_3;$$
$$\Gamma_{wedge} := \partial\Omega \cap \partial\Lambda \equiv P_3P_4.$$

Note the boundary conditions on $\partial\Omega$:

$$\begin{cases} \rho(|\nabla\varphi|^2, \varphi)\nabla\varphi \cdot \upsilon = \rho(|\nabla\varphi_1|^2, \varphi_1)\nabla\varphi_1 \cdot \upsilon \\ \varphi = \varphi_1 \end{cases} \quad \text{on } \Gamma_{shock}; \tag{2-22}$$

$$\varphi = \varphi_2 \quad \text{on } \Gamma_{sonic}; \tag{2-23}$$

$$\varphi_\upsilon = 0 \quad \text{on } \Gamma_{wedge}; \tag{2-24}$$

$$\varphi_{\upsilon=0} \quad \text{on } \Gamma_{symm}. \tag{2-25}$$

We should be careful here, if we want the solution φ in Ω to be a part of global solution to Problem 2.3, i.e., φ satisfies the equation in distribution sense in the domain Λ, we need:

$$\nabla(\varphi - \varphi_2) \cdot \upsilon|_{\Gamma_{sonic}} = 0 \tag{2-26}$$

In fact this condition is proposed for matching φ with state (2), and we need to show φ

is at least C^1 with $\nabla(\varphi - \varphi_2) = 0$ across Γ_{sonic} to obtain (2-26).

Then the problem can be reformulated as the free boundary problem:

Problem 2.4 (Free Boundary Problem)**.** *Seek a global solution φ and a free boundary $\Gamma_{sonic} = \{\xi = g(\eta)\}$ for satisfying:*

(1) free boundary function $g \in C^{1,\alpha}$

(2) φ satisfies the free boundary condition (2-22) on the Γ_{sonic}

(3) define the free boundary

$$\Omega^+ = \{\xi > g(\eta)\} \cap D \tag{2-27}$$

then $\varphi \in C^{1,\alpha}(\overline{\Omega^+}) \cap C^2(\Omega^+)$ solves (2-9) and (2-10) in Ω^+, and also satisfies the equations (2-23), (2-24), (2-25) and the conormal boundary condition (2-26) on Γ_{sonic}.

Remark 2.5. *The definition of Ω^+ suggests the free boundary area is determined by the level set $\varphi = \varphi_1$. The boundary condition on Γ_{symm} ensures $g'(0) = 0$, which implies the orthogonality of the free boundary with ξ-axis. It suggests thus we cannot apply this free boundary technique to asymmetric wedge case.*

Chapter 3 Global structure of regular reflection for potential flow

In this chapter, I first describe the simplest case of shock reflection problem which occurs when the wedge angle θ_w is $\pi/2$. In this case, the reflection problem becomes the normal reflection problem and thus the incident shock normally reflected with a resulting plane reflected shock. Second half of this chapter concentrates on the global theory established by Chen-Feldman[9][10] and Bea-Chen-Feldman[11] for solving Problem 2.4. Moreover, the constructed global solution tends to converge to normal reflected case, i.e., guarantee the stability of solution.

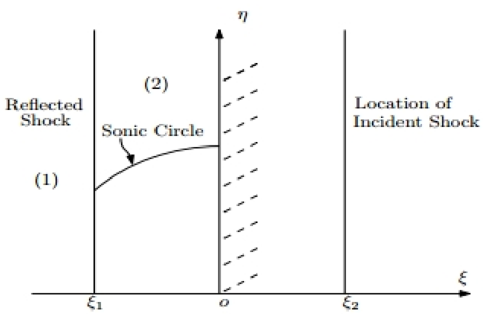

Figure 3.1 Normal Reflection

3.1 Normal reflection

When the wedge becomes flat, i.e., $\theta_\omega = \frac{\pi}{2}$, the reflection becomes the normal reflection which is the simplest case (see Fig 3.1). In this case, the incident shock normally reflects, the reflected shock is also a plane at $\xi = \bar{\xi}$, and $(u_2, v_2, \rho_2) = (0, 0, \overline{\rho_2})$, where

$$\bar{\xi} = -\frac{\rho_1 u_1}{\bar{\rho_2} - \rho_1} < 0, \quad \text{with} \quad u_1 = \sqrt{\frac{(p_2 - p_1)(\rho_2 - \rho_1)}{\rho_1 \rho_2}}$$

and $\bar{\rho_2} > \rho_1$ is the unique solution of the Bernoulli law

$$\bar{\rho_2}^{\gamma-1} = \rho_1^{\gamma-1} + \frac{1}{2} u_1^2 + \frac{\rho_1 u_1^2}{\bar{\rho_2} - \rho_1}.$$

Then state (1) has form (2-16) for $\xi < \bar{\xi}$, state (2) has the form:

$$\varphi_2(\xi,\eta) = -\frac{1}{2}(\xi^2+\eta^2) + u_1(\bar{\xi}-\xi_0) \quad \text{for} \quad \xi \in (\bar{\xi}, 0),$$

and the reflected shock $\xi = \bar{\xi}$ actually satisfies the entropy condition: $\bar{\rho}_2 > \rho_1$. Moreover, it can be shown that $|\bar{\xi}| < \bar{c}_2 := c(\bar{\rho}_2)$. Then $\sqrt{\xi^2+\eta^2} = \bar{c}_2$ is the sonic circle, and the subsonic region of state (2) is $B_{\bar{c}_2}(0) \cap \{\bar{\xi} < \xi < 0\}$, and is supersonic outside the sonic circle (see Fig 3.1). That is, the normal reflection solution is uniquely determined.

3.2 Existence and stability of global solution

In [9][10], Chen-Feldman established a rigorous mathematical method for solving free boundary Problem 2.4 and developed the existence of global solution φ for potential flow when the wedge angle θ_ω is large with the stability for solution which converges to the unique solution of the normal reflection when wedge angle tends to $\pi/2$. The corresponding structure of solution is presented in Chapter 2.3 as following Fig 2.2. Here I grasp some primary techniques as well as all important properties of the solution.

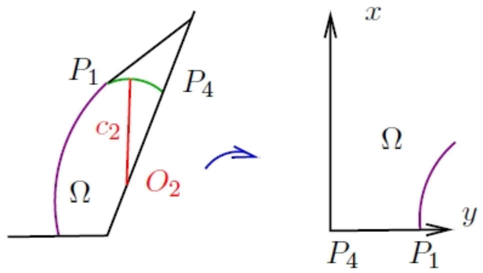

Figure 3.2 Flatten sonic arc

First we introduce the polar coordinates (r,θ) with the polar center $(u_2, u_2\tan\theta_\omega)$ of the pseudo-sonic circle of state (2), that is

$$\xi - u_2 = r\cos\theta, \quad \eta - u_2\tan\theta_\omega = r\sin\theta \tag{3-1}$$

For each ϵ-neighborhood of the pseudo-sonic circle P_1P_4 within Ω, we denote it as

$$\Omega_\epsilon := \Omega \cap \{(r, \theta) : 0 < c_2 - r < \epsilon\}, \quad \epsilon \in (0, c_2).$$

More specifically, in the Ω_ϵ, we use the coordinates:

$$x = c_2 - r, \quad y = \theta - \theta_\omega \tag{3-2}$$

This implies that $\Omega_\epsilon \subset \{0 < x < \epsilon, y > 0\}$ and $P_1P_4 \subset \{x = 0, y > 0\}$.(See Fig 3.2) Also we note we have introduced the following notation for various parts of $\partial\Omega$:

$$\begin{aligned}
&\Gamma_{sonic} := \partial\Omega \cap \partial B_{c_2}(u_2, u_2 \tan\theta_\omega) \equiv P_1P_4;\\
&\Gamma_{shock} := P_1P_2;\\
&\Gamma_{wedge} := \partial\Omega \cap \partial\Lambda \equiv P_3P_4.
\end{aligned}$$

The idea proposed by Chen-Feldman is to regard this reflection problem as free boundary problem,i.e., to solve Problem 2.4.

Theorem 3.1 (Chen-Feldman[9][10]). *There exist $\theta_c = \theta_c(\rho_0, \rho_1, \gamma) \in (0, \frac{\pi}{2})$ and $\alpha = \alpha(\rho_0, \rho_1, \gamma) \in (0, 1/2)$ such that, when $\theta_\omega \in [\theta_c, \frac{\pi}{2})$, there exists a global self-similar solution:*

$$\Phi(\boldsymbol{x}, t) = t\varphi(\frac{\boldsymbol{x}}{t}) + \frac{|\boldsymbol{x}|^2}{2t} \quad for \frac{\boldsymbol{x}}{t} \in \Lambda, t > 0,$$

with

$$\rho(\boldsymbol{x}, t) = (\rho_0^{\gamma-1} - (\gamma - 1)(\Phi_t + \frac{1}{2}|\nabla_{\boldsymbol{x}}\Phi|^2))^{\frac{1}{\gamma-1}}$$

of Problem 2.1 (also, Problem 2.3) for shock reflection by the wedge, which satisfies *that, for* $(\xi, \eta) = \frac{\boldsymbol{x}}{t}$,

$$\begin{aligned} &\varphi \in C^{0,1}(\Lambda), \\ &\varphi \in C^{\infty}(\Omega) \cap C^{1,\alpha}(\overline{\Omega}), \\ &\varphi = \begin{cases} \varphi_0 & \text{for } \xi > \xi_0 \text{ and } \eta > \xi \tan\theta_\omega \\ \varphi_1 & \text{for } \xi < \xi_0 \text{ and above the reflection shock } P_0P_1P_2 \\ \varphi_2 & \text{in } P_0P_1P_4 \end{cases} \end{aligned} \tag{3-3}$$

Moreover,

1. *equation (2-9) is elliptic in* Ω*;*
2. $\varphi_2 \leq \varphi \leq \varphi_1$ *in* Ω*;*
3. *there exist* $\epsilon_0 \in (0, \frac{c_2}{2})$ *such that* $\varphi \in C^{1,1}(\overline{\Omega_{\epsilon_0}}) \cap C^2(\overline{\Omega_{\epsilon_0}} \backslash \overline{\Gamma_{sonic}})$*; moreover, in the coordinates (3-2) we define,*

$$||\varphi - \varphi_2||^{(par)}_{2,0,\Omega_{\epsilon_0}} := \sum_{0 \leq k+l \leq 2} \sup(x^{k+l/2-2}|\partial_x^k \partial_y^l (\varphi - \varphi_2)(x,y)|) < \infty \tag{3-4}$$

4. *there exist* $\omega > 0$ *and a function* $y = \hat{f}(x)$ *such that, in the coordinate (3-2) we have,*

$$\begin{aligned} &\Omega_{\epsilon_0} = \{(x,y) : x \in (0, \epsilon_0), 0 < y < \hat{f}(x)\}, \\ &\Gamma_{shock} \cap \{0 \leq x \leq \epsilon_0\} = \{(x,y) : x \in (0, \epsilon_0), y = \hat{f}(x)\}, \end{aligned} \tag{3-5}$$

and

$$||\hat{f}||_{C^{1,1}([0,\epsilon_0])} < \infty, \quad \frac{d\hat{f}}{dx} \geq \omega > 0 \quad \text{for} \quad 0 < x < \epsilon_0 \tag{3-6}$$

5. *the reflected shock* $P_0P_1P_2$ *is* C^2 *at* P_1 *and* C^∞ *except* P_1*;*
6. *there exists* $\varrho_0 > 0$ *so that, in the coordinates (3-2),*

$$|\partial_x(\varphi - \varphi_2)(x,y)| \leq \frac{2 - \varrho_0}{\gamma + 1} x \qquad \text{in} \quad \Omega_{\epsilon_0}, \tag{3-7}$$

Moreover, it is to say the solution φ is stable in terms of the wedge angle in $W^{1,1}_{loc}$ and converges in $W^{1,1}_{loc}$ to the uniquely determined solution of the normal reflection as $\theta_w \to \pi/2$

The existence of state (2) of form (2-21) with constant velocity $(u_2, u_2 \tan\theta_\omega)$, $u_2 > 0$, and constant density $\rho_2 > \rho_1$, satisfying Rankine-Hugonior jump condition (2-14) on P_0P_1 is shown in [10][section 3] for $\theta_\omega \in [\theta_c, \frac{\pi}{2})$. The existence of a solution φ of Problem 2.3, satisfying (2-22) and property 3 follows from [10][Main Theorem]. Property 4 follows from (5.7) and (5.25)-(5.27) in [10] and the fact that $\varphi - \varphi_2 \in \Theta$.

Remark 3.2. *We should note the norm imposed in (3-4) with the finite estimate further shows the fact: the potential flow equation (2-9) and (2-10) coincides with the full Euler equation in a subdomain of Ω bounded by the streamline of pseudo-velocity field passing through P_1, sonic arc and wedge.*

Chapter 4 Estimation for solution near the sonic arc

One of the primary difficulties for constructing the global existence theory is that the ellipticity condition (2-12) is hard to handle with. The second difficulty is that the ellipticity degenerates at the sonic circle P_1P_4, with the non-variational structure and nonlinearity for the equation behaving near sonic arc area. However, we need to match the solution φ with φ_2 at least in C^1, with both Dirichlet and conormal conditions.

In this chapter, we will show the conormal condition follows automatically from the structure of elliptic degeneracy of (2-9) on P_1P_4 for φ. For achieving this, we first adopt a proper cutoff that depends on the distance to the sonic circle to modify equation (2-12) and we have the modified equation to be elliptic in Ω with elliptic degeneracy on P_1P_4 (see Section 4.1 and 4.2). Then I give a proof to show φ is in fact $C^{1,1}$ up to the boundary, especially $|D(\varphi - \varphi_2)| \leq Cx$, and confirm the cornormal condition on the sonic arc (see Section 4.3). At last, I point out the ellipticity cutoff in fact can be removed off since we can prove the precise gradient estimate $|u_x| < \frac{(2-\beta)x}{\gamma+1}$ for $\beta > 0$ (see Section 4.3).

Remark 4.1. *Noting here, for a regular reflection configuration, if we start from the full Euler flow we can also prove the exact behavior of solutions in $C^{1,1}$ across the pseudo-sonic circle which coincides with the potential flow. This means, both of the nonlinear systems actually behavior the same in a physically significant domain near the pseudo-sonic circle. However, we can prove some fundamental properties of solutions for the potential flow. For example, the optimal regularity of solutions across the pseudo-sonic circle and at the degenerate point where the pseudo-sonic circle meets the reflected shock particular for the potential flow (see Chapter 5).*

First I prove an important property which is extremely useful in our following analysis. Introduce a new variable

$$\psi = \varphi - \varphi_2 = \phi - \phi_2$$

while will play a core role.

Proposition 4.2. *$\varphi \geq \varphi_2$, i.e., $\psi \geq 0$ in Ω*

Remark 4.3. *This property (I also point it out in section 3.2) follows from Proposition 7.1 and Section 9 in [10], which assert that $\varphi - \varphi_2 \in \Theta$, where the set Θ defined by (5.15) in [10]. However, I will give a general and distinct proof for large-angle wedge.*

Proof for Proposition 4.2. Since $\psi = \varphi - \varphi_2 = \phi - \phi_2$, we have the homogeneous equation in Ω in term of ψ, i.e. $\Sigma A_{ij}\psi_{ij} = 0$ in Ω.

Now we rewrite the boundary conditions in terms of ψ. First the boundary condition on $\Gamma_{symm} = P_2P_3$ indicates in (2-25), here the normal vector $\upsilon = (0,1)$ in (ξ,η)-coordinates. By definition, $D\phi = D\varphi + (\xi,\eta)'$ and $\eta = 0$ on P_2P_3 . Thus

$$D\phi \cdot \upsilon = D\varphi \cdot \upsilon + (\xi, 0) \cdot (0,1)' = 0$$

from (2-25). Note $D\phi_2 = (u_2, u_2 tan\theta_w)$, we have

$$\psi_\upsilon = (D\phi - D\phi_2) \cdot \upsilon = -u_2 < 0 \tag{4-1}$$

Second on $\Gamma_{shock} = P_1P_2$ the Rankine-Hugoniot jump condition (2-22) can be rewritten as homogeneous jump condition $a_1\psi_\xi + a_2\psi_\eta + a_3\psi = 0$ with $a_3 < 0$, and $(a_1, a_2) \cdot \upsilon_{shock} \geq \lambda > 0$, here λ is a positive constant and υ_{shock} is the normal vector with respect to the reflected shock. If $\psi = 0$, this homogeneous equation also equal to 0. Third on $\Gamma_{sonic} = P_1P_4$ we have $\psi = 0$ and on $\Gamma_{wedge} = P_3P_4$ we have $\psi_\upsilon = 0$.

Therefore, ψ is supersolution of homogeneous problem and 0 is solution of that problem. By maximum principle, $\psi \geq 0$ as desired. $\sharp$

4.1 Reformulation of ψ

Now we can explicitly calculate that ψ satisfies the following equation in Ω:

$$\begin{aligned}(c^2(D\psi,\psi,\xi,\eta) - (\psi_\xi - \xi)^2)\psi_{\xi\xi} + (c^2(D\psi,\psi,\xi,\eta) - (\psi_\eta - \eta)^2)\psi_{\eta\eta} \\ -2(\psi_\xi - \xi)(\psi_\eta - \eta)\psi_{\xi\eta} = 0\end{aligned} \tag{4-2}$$

and the expressions of the density and sound speed in Ω in terms of ψ are

$$\rho(D\psi,\psi,\xi,\eta)=(\rho_2^{\gamma-1}+\xi\psi_\xi+\eta\psi_\eta-\frac{1}{2}|D\psi|^2-\psi)^{\frac{1}{\gamma-1}},$$

$$c^2(D\psi,\psi,\xi,\eta)=c_2^2+(\gamma-1)(\xi\psi_\xi+\eta\psi_\eta-\frac{1}{2}|D\psi|^2-\psi),$$

where ρ_2 is the density of state (2).

Moreover, we can rewrite (4-2) in the form

$$T_1+T_2+T_3+T_4=0,$$

where

$$\begin{aligned}
T_1&=(c^2(D\psi,\psi,\xi,\eta)-(\xi^2+\eta^2))\Delta\psi,\\
T_2&=\eta^2_{\xi\xi}+\xi^2_{\eta\eta}-2\xi\eta\psi_{\xi\eta},\\
T_3&=2(\xi\psi_\xi\psi_{\xi\xi}+(\xi\psi_\eta+\eta\psi_\xi)\psi_{\xi\eta}+\eta\psi_\eta\psi_{\eta\eta}),\\
T_4&=-\frac{1}{2}(\psi_\xi(|D\psi|^2)_\xi+\psi_\eta(|D\psi|^2)_\eta).
\end{aligned}$$

In the polar coordinates (r,θ) with $r=\sqrt{\xi^2+\eta^2}$, note that $\psi_\xi=\frac{\xi}{r}\psi_r-\frac{\eta}{r^2}\psi_\theta$ and $\psi_\eta=\frac{\eta}{r}\psi_r+\frac{\xi}{r^2}\psi_\theta$. Therefore ψ satisfies

$$\begin{aligned}
(c^2-(\psi_r-r)^2)\psi_{rr}-\frac{2}{r^2}(\psi_r-r)\psi_\theta\psi_{r\theta}+\frac{1}{r^2}(c^2-\frac{1}{r^2}\psi_\theta^2)\psi_{\theta\theta}+&\\
\frac{c^2}{r}\psi_r+\frac{1}{r^3}(\psi_r-2r)\psi_\theta^2=0&
\end{aligned}\tag{4-3}$$

with sonic speed $c^2=(\gamma-1)(\rho_2^{\gamma-1}-\psi+r\psi_r-\frac{1}{2}(\psi_r^2+\frac{1}{r^2}\psi_\theta^2))$. Note that, in the polar coordinates, T_1,T_2,T_3,T_4 have the expressions as:

$$\begin{aligned}
T_1&=(c_2^2-r^2+(\gamma-1)(r\psi_r-\frac{1}{2}|D\psi|^2-\psi))\Delta\psi,\\
T_2&=\psi_{\theta\theta}+r\psi_r,\\
T_3&=2r\psi_r\psi_{rr}+\frac{2}{r}\psi_\theta\psi_{r\theta}-\frac{2}{r^2}\psi_\theta^2=r(|D\psi|^2)_r,\\
T_4&=-\frac{1}{2}(\psi_r(|D\psi|^2)_r+\frac{1}{r^2}\psi_\theta(|D\psi|^2)_\theta).
\end{aligned}$$

Here $|D\psi|^2 = \psi_r^2 + \frac{1}{r^2}\psi_\theta^2$ and $\Delta\psi$ is taken with respect to (r,θ), i.e. $\Delta\psi = \psi_{rr} + \frac{1}{r^2}\psi_{\theta\theta} + \frac{1}{r}\psi_r$.

From these relations, the self-similar potential flow equation can thus be rewritten for ψ in (x,y)-coordinates (by (3-2)) as

$$(2x-(\gamma+1)\psi_x+O_1)\psi_{xx}+O_2\psi_{xy}+(\frac{1}{c_2}+O_3)\psi_{yy}-(1+O_4)\psi_x+O_5\psi_y=0 \quad (4\text{-}4)$$

in $\Omega \subset \{x>0\}$. Where

$$\begin{aligned}
O_1(D\psi,\psi,x) &= -\frac{x^2}{c_2}+\frac{\gamma+1}{2c_2}(2x-\psi_x)\psi_x-\frac{\gamma-1}{c_2}(\psi+\frac{1}{2(c_2-x)^2}\psi_y^2);\\
O_2(D\psi,\psi,x) &= -\frac{2}{c_2(c_2-x)^2}(\psi_x+c_2-x)\psi_y;\\
O_3(D\psi,\psi,x) &= \frac{1}{c_2(c_2-x)^2}(x(2c_2-x)-(\gamma-1)(\psi+(c_2-x)\psi_x+\frac{1}{2}\psi_x^2)\\
&\quad -\frac{\gamma+1}{2(c_2-x)^2}\psi_y^2);\\
O_4(D\psi,\psi,x) &= \frac{1}{c_2-x}(x-\frac{\gamma-1}{c_2}(\psi+(c_2-x)\psi_x+\frac{1}{2}\psi_x^2\\
&\quad +\frac{(\gamma+1)\psi_y^2}{2(\gamma-1)(c_2-x)^2}));\\
O_5(D\psi,\psi,x) &= -\frac{2}{c_2(c_2-x)^3}(\psi_x+c_2-x)\psi_y.
\end{aligned}$$

The terms $O_k(D\psi,\psi,x)$ are small perturbations of the leading terms of (4-4) if the function ψ is small in appropriate norm (note we have Dirichlet boundary condition (2-23) on sonic circle P_1P_4). In order to see this, we turn to the following properties: For any $(p,z,x)\in R^2\times R\times(0,c_2/2)$ with $|p|<1$,

$$\begin{aligned}
&|O_1(p,z,x)| \le C(|p|^2+|z|+|x^2|),\\
&|O_3(p,z,x)|+|(O_4(p,z,x))| \le C(|p|+|z|+|x|),\\
&|O_2(p,z,x)|+|(O_5(p,z,x))| \le C(|p|+|x|+1)|p|.
\end{aligned}$$

For convenient, we can drop terms $O_i, i=1,\ldots,5$, from (4-4), we obtain the transonic small disturbance equation:

$$(2x - (\gamma + 1)\psi_x)\psi_{xx} + \frac{1}{c_2}\psi_{yy} - \psi_x = 0 \tag{4-5}$$

Thus the full equation is homogeneous. In (x, y)-coordinates, we have proved in Proposition 4.2 that

$$\psi > 0 \quad in \quad \Omega$$
$$\psi = 0 \quad on \quad \Gamma_{sonic} = \partial\Omega \cap \{x = 0\}$$

Equation (4-5) is elliptic in $\{x > 0\}$ if

$$\psi_x < \frac{2x}{\gamma + 1} \tag{4-6}$$

Proposition 4.4. *Assume (4-6) holds then*

$$\psi \le Cx^2 \quad in \quad \Omega \tag{4-7}$$

and C is a positive constant.

Proof for Proposition 4.4 Set $w(x, y) := \frac{A}{2(\gamma+1)}x^2$, with $A > 1$. Denote $N(\psi)$ to be the left-hand side of equation (4-5). We first show that w is a supersolution of (4-5).

1) Substitute $w(x, y)$, we have $N(w) < (2x - x)\frac{A}{\gamma+1} - \frac{A}{\gamma+1}x \equiv 0 = N(\psi)$;

On boundaries, we can note:

2) For Γ_{wedge}, i.e. $y = 0$, we have $w_\nu = w_y = 0 = \psi_\nu$;

3) For Γ_{shock}, i.e. the right plot in Figure 4, we have boundary condition $M(\psi) = a_1\psi_x + a_2\psi_y + a_3\psi = 0$ with $a_k \le -\delta < 0$, $|a_k| \le C, k = 1, 2, 3$, here δ and C are positive constants. And $(a_1, a_2) \cdot \upsilon_{shock} \ge \lambda > 0$, here λ is a positive constant and υ_{shock} is the normal vector with respect to the reflected shock.Thus plug in w we have $M(w) = a_1\frac{A}{\gamma+1}x + a_3\frac{A}{2(\gamma+1)}x^2 < 0 = M(\psi)$

4) For Γ_{sonic}, i.e. $x = 0$, we have $w = \psi = 0$;

5) For $\{x = \epsilon\} \cap \partial\Omega_\epsilon$ with $\Omega_\epsilon = \Omega \cap \{0 < x < \epsilon\}$. Note Ω is a bounded area, ψ

is thus also bounded, that is $||\psi||_{C^1(\Omega)} \leq C_1$. In particular, we assume $||\psi||_{C^1(\Omega)} \leq C\delta$ for $\theta_w \in (\frac{\pi}{2} - \delta, \frac{\pi}{2})$. Choose $w = \frac{A\epsilon^2}{2(\gamma+1)} = C\delta \geq \psi$, i.e. $A = \frac{2C\delta(\gamma+1)}{\epsilon^2}$, can guarantee $w \geq \psi$ on $\{x = \epsilon\} \cap \partial\Omega_\epsilon$.

By Maximum Principle and the 5 points above, we can assert $w \geq \psi$ and thus $\psi \leq Cx^2$ as desired. $\sharp$

4.2 Estimation with the ellipticity cutoff

Taking into account the ”small” terms to be added to equation (4-5) (to recover (4-4)), we need to make the stronger estimate $|\psi_x| \leq \frac{4x}{3(\gamma+1)}$. In fact this conjecture exactly holds (I will talk about it in section 4.4) and thus Proposition 4.4 follows. We can modify equation (2-9) in Ω by a proper cutoff that depends on the distance to the sonic circle, so that the original and modified equations coincide for φ satisfying $|\psi_x| \leq \frac{4x}{3(\gamma+1)}$, and the modified equation is elliptic in Ω with elliptic degeneracy on P_1P_4.

Introduce the function $\zeta \in C^\infty(R)$ satisfy

$$\zeta(s) = \begin{cases} s & \text{if } |s| < 4/(3(\gamma+1)) \\ \frac{5sign(s)}{3(\gamma+1)} & \text{if } |s| > 2/(\gamma+1) \end{cases} \tag{4-8}$$

thus

$$\zeta'(s) \geq 0, \quad \zeta(-s) = -\zeta(s) \text{ on } R, \text{ and } \zeta''(s) \leq 0 \text{ on } \{s \geq 0\} \tag{4-9}$$

Obviously, such a smooth function $\zeta \in C^\infty(R)$ exists.

Introduce the notations T_1' and T_3' for the corresponding T_1 and T_3 respectively as

$$\begin{aligned} T_1' = \ & (c_2^2 - r^2 + (\gamma+1)r(c_2 - r)\zeta(\tfrac{\xi\psi_\xi + \eta\psi_\eta}{r(c_2-r)}) - (\gamma-1)(\tfrac{1}{2}|D\psi|^2 + \psi))\Delta\psi \\ T_3' = \ & 2(\tfrac{\xi}{r}(c_2 - r)\zeta(\tfrac{\xi\psi_\xi + \eta\psi_\eta}{r(c_2-r)}) - \tfrac{\eta}{r^2}(\xi\psi_\eta - \eta\psi_\xi))(\xi\psi_{\xi\xi} + \eta\psi_{\xi\eta}) \\ & +2(\tfrac{\eta}{r}(c_2 - r)\zeta(\tfrac{\xi\psi_\xi + \eta\psi_\eta}{r(c_2-r)}) + \tfrac{\xi}{r^2}(\xi\psi_\eta - \eta\psi_\xi))(\xi\psi_{\xi\eta} + \eta\psi_{\eta\eta}) \end{aligned}$$

The modified equation is defined as

$$T_1' + T_2 + T_3' + T_4 = 0 \tag{4-10}$$

From the definition of ζ, the modified equation (4-10) coincides with the original (4-2) if

$$\left|\frac{\xi\psi_\xi + \eta\psi_\eta}{r(c_2 - r)}\right| < \frac{4}{3(\gamma+1)},$$

i.e., if $|\psi_x| < \frac{4x}{3(\gamma+1)}$ in the (x, y)-coordinates as we assumed above.

Now we can write (4-10) in the (x, y)-coordinates. Calculate T_1' and T_3' in the polar coordinates as

$$T_1' = (c_2^2 - r^2 + (\gamma - 1)(r(c_2 - r)\zeta(\frac{\psi_r}{c_2 - r}) - \frac{1}{2}|D\psi|^2 - \psi))\Delta\psi$$
$$T_3' = 2r(c_2 - r)\zeta(\frac{\psi_r}{c_2 - r})\psi_{rr} + \frac{2}{r}\psi_\theta\psi_{r\theta} - \frac{2}{r^2}\psi_\theta^2$$

and transform (4-10) to the form

$$\begin{aligned}(2x - (\gamma+1)x\zeta(\frac{\psi_x}{x}) + O_1')\psi_{xx} + O_2'\psi_{xy} + (\frac{1}{c_2} + O_3')\psi_{yy} \\ -(1 + O_4')\psi_x + O_5'\psi_y = 0\end{aligned} \tag{4-11}$$

here we have the estimation for the small terms

$$|O_1'(p, x, y)| \le C|x|^{3/2}, \quad |O_k'(p, x, y)| \le C|x| \text{ for } k = 2, 3, 4, 5 \tag{4-12}$$

for all $p \in R^2$. Now we can do the estimation more precisely.

Theorem 4.5. *The solution $\psi \in C(\overline{\Omega}) \cap C^1(\overline{\Omega}\backslash\overline{\Gamma_{sonic}}) \cap C^2(\Omega)$ satisfies*

$$\psi \le \frac{2}{3(\gamma+1)}x^2 \quad \text{in} \quad \Omega_\epsilon \tag{4-13}$$

here $\Omega_\epsilon = \Omega \cap \{0 < x < \epsilon\}$.

Proof for Theorem 4.5 First rewrite equation (42) as $N_1(\psi) + N_2(\psi) = 0$, with

$$N_1(\psi) = (-(\gamma+1)x\zeta(\frac{\psi_x}{x}) + 2x)\psi_{xx} + \frac{1}{c_2}\psi_{yy} - \psi_x, \tag{4-14}$$
$$N_2(\psi) = O_1'\psi_{xx} + O_2'\psi_{xy} + O_3'\psi_{yy} + O_4'\psi_x + O_5'\psi_y \tag{4-15}$$

Set $w(x,y) := \frac{A}{2(\gamma+1)}x^2$, with $A \geq \frac{4}{3}$. As in Proposition 4.4, we easily notice w is a supersolution for (4-11). Calculate $w_x = \frac{A}{\gamma+1}x$, $w_{xx} = \frac{A}{\gamma+1}$ and

$$\frac{w_x}{x} = \frac{A}{\gamma+1} \geq \frac{4}{3(\gamma+1)}$$

and thus

$$\zeta(\frac{w_x}{x}) \geq \frac{4}{3(\gamma+1)} > 0.$$

Now

$$N_1(\psi) \leq \left(2x - (\gamma+1)\frac{4}{2(\gamma+1)}x\right)\frac{A}{\gamma+1} - \frac{A}{\gamma+1}x = -\frac{A}{3(\gamma+1)}x \leq -\frac{4}{9(\gamma+1)}$$

Using equation (43), we have

$$|N_2(w)| = \left|\frac{A}{\gamma+1}O_1'(Dw,x,y) + \frac{Ax}{\gamma+1}O_2'(Dw,x,y)\right| \leq Cx^{\frac{3}{2}} \leq C\epsilon^{\frac{1}{2}}x,$$

here the last inequality holds since $x \in (0,\epsilon)$ in Ω_ϵ. Thus,

$$N_1(w) + N_2(w) \leq -\frac{4}{9(\gamma+1)}x + C\epsilon^{\frac{1}{2}}x < 0,$$

where the last inequality holds if $\epsilon \in (0, (\frac{4}{9C(\gamma+1)})^2)$. Therefore $N(w) = N_1(w) + N_2(w) < 0 = N(\psi)$ in Ω_ϵ

On boundaries, we also argue that w is a supersolution of $M(\psi) = a_1\psi_x + a_2\psi_y + a_3\psi = 0$ on Γ_{shock} in the same way as Proposition 4.4. Moreover, the boundary conditions on Γ_{wedge}, Γ_{sonic} and $\{x = \epsilon\} \cap \partial\Omega_\epsilon$ also follows as proved in Proposition 4.4.

By Maximum Principle, we can conclude $w \geq \psi$ and thus $\psi \leq \frac{2}{3(\gamma+1)}x^2$ as desired. ♯

4.3 Regularity near sonic arc

From Proposition 4.2 and Theorem 4.5, we get

$$0 \leq \psi \leq Cx^2. \tag{4-16}$$

From monotonocities of $\psi = \varphi - \varphi_2$ near sonic arc, $0 \leq \psi_x \leq Cx$, $|\psi_y| \leq Cx$. And thus we can control the coefficients of equation. Comparing with equation (4-5), for simplicity we consider

$$xu_{xx} + u_{yy} - \alpha u_x = 0 \text{ in } \{x > 0\} \tag{4-17}$$

and since (4-16) holds we can further assume $|u| \leq Cx^2$.

Assume $x_0 = 2d$, consider the rectangle

$$Q_d(x_0, y_0) = \{(x, y)||x - x_0| < d, |y - y_0| \leq \sqrt{d}\}.$$

As showed in Figure 4.1, $Q_d(x_0, y_0) \subset \{x > 0\}$.

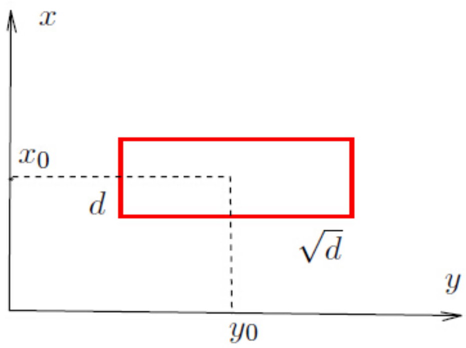

Figure 4.1 Change Variables

Proposition 4.6. *The solution u of (46) is $C^{1,1}$ up to sonic arc $\{x{=}0\}$, i.e.*

$$\begin{gathered} |u_x| \leq Cx, \quad |u_y| \leq Cx^{3/2}, \\ |u_{xx}| \leq C, \quad |u_{xy}| \leq Cx^{1/2}, \quad |u_{yy}| \leq Cx. \end{gathered} \tag{4-18}$$

Proof for Proposition 4.6 First change the variables (x, y) to (X, Y) as

$$X = \frac{x - x_0}{d}, \quad Y = \frac{y - y_0}{\sqrt{d}}.$$

we map the rectangle $Q_d(x_0, y_0)$ to the unit square $Q_1(0, 0)$.

Then define $z(X,Y)$ on $Q_1(0,0)$ by

$$z(X,Y) \equiv z(\frac{x-x_0}{d}, \frac{y-y_0}{\sqrt{d}}) = \frac{1}{d^2}u(x,y),$$

and calculate $u_x = dz_X, u_y = d^{3/2}z_Y, u_{xx} = z_{XX}, u_{yy} = dz_{YY}$ and translate the equation (46) for u into:

$$(2+X)z_{XX} + z_{YY} - \alpha z_X = 0, \tag{4-19}$$

we see (48) is uniformly elliptic in $Q_1(0,0)$. Thus for Holder norm,

$$||z||_{C^{2,\alpha}(Q_{1/2})} \leq C||z||_{L^\infty(Q_1)} \leq \widehat{C}.$$

Rewrite this in terms of $u(x,y)$ at $(x,y) = (x_0,y_0)$, we can get $C^{1,1}$ regularity (4-18) as desired. $\sharp$

Essentially this scaling technique involves linearization near $w = \frac{1}{2(\gamma+1)}x^2$ and it is showed to control nonlinear equation up to $C^{1,1}$.

4.4 Ellipticity and Removal of cutoff

Relation (4-18) tells us $|\psi_x| < Cx$ for some possible large but uniform constant C, how ever it is not enough for removal of ellipticity cutoff. As pointed out in section 4.2, we need at least $|\psi_x| < Cx$ with $C = \frac{4}{2(\gamma+1)}$ and thus we have to do more precise estimation for $|\psi_x|$.

Similar to the growth estimates,estimate of ψ_x from above and from below come from the different reasons.

Proposition 4.7 (estimation from above)**.** *$\psi_x \leq \frac{k}{\gamma+1}x$ holds for any $k \in (1,2)$ in Ω_ϵ*

Remark 4.8. *The idea here is that estimate from above is local near the sonic arc by comparison function. We should note if staying sufficiently close to the sonic arc we can get $\psi_x < Ax$ for any $A > \frac{1}{\gamma+1}$, but cannot make $A < \frac{1}{\gamma+1}$. Thus in the choice of cutoff function we use $\frac{4}{3}\frac{1}{\gamma+1}$ (it is also necessary to keep cutoff at smaller than $2\frac{1}{(\gamma+1)}$ to have* *the elliptic in (4-4)).*

Proof for Proposition 4.7 Here I only prove for $k = \frac{4}{3}$, and the proof for general situations is exact the same if we adjust the cutoff function. Denote $A = \frac{4}{3(\gamma+1)}$ and define a function

$$v := Ax - \psi_x \tag{4-20}$$

we only need to show that $v \geq 0$ in a neighborhood of the sonic arc.

For that, we differentiate equations (4-11)-(4-12) and get the following equation for v:

$$a_{11}v_{xx} + a_{12}v_{xy} + a_{22}v_{yy} + bv_x + cv = -A((\gamma+1)A - 1) + S(x, y) \tag{4-21}$$

with coefficients

$$b(x, y) = 1 - (\gamma+1)(\zeta(A - \frac{v}{x}) + \zeta'(A - \frac{v}{x})(\frac{v}{x} - v_x - A)), \tag{4-22}$$

$$c(x, y) = (\gamma+1)\frac{A}{x}(\zeta'(A - \frac{v}{x}) - \int_o^1 \zeta'(A - s\frac{v}{x})ds). \tag{4-23}$$

Since the equation (4-21) has right-hand side with the main term to be negative if $A > 1/(\gamma+1)$ (this explains why we need to impose the restriction as mentioned on Remark 4.8), it means that v is a supersolution of the homogeneous equation with the following facts hold:

1) The left hand of equation (4-21) is elliptic in Ω_ϵ and uniformly elliptic on compact subsets of $\overline{\Omega_\epsilon}\backslash\{x = 0\}$.

On boundaries, we need to use estimate (4-18) in Ω_ϵ

2) since $|\psi_x| \leq Cx$, $v = 0$ on $\partial\Omega_\epsilon \cap \{x = 0\}$

3) since $|\psi| \leq Cx^2$, $|\psi_y| \leq Cx^{3/2}$ and on shock we have $a_1\psi_x + a_2\psi_y + a_3 = 0$, we have the estimate $|\psi_x| \leq C_1(|\psi_y| + |\psi|) \leq C_2x^{\leq}3/2$ and hence $|\psi_x| < Ax$ on $\Gamma_{shock} \cap \{0 < x < \epsilon\}$. Thus

$$v \geq 0 \quad \text{on} \quad \Gamma_{shock} \cap \{0 < x < \epsilon\}$$

4) note on Γ_{wedge} the boundary condition (2-24), and on (x, y)-coordinates it is

$$\psi_y = 0 \quad \text{on} \quad \{0 < x < \epsilon, y = 0\}.$$

Since ψ is C^2 up to Γ_{wedge}, we have $\psi_{xy} = 0$, which implies

$$v_y = 0 \quad \text{on} \quad \Gamma_{wedge} \cap \{0 < x < \epsilon\}.$$

5) furthermore, since $|\psi_x| \leq A\epsilon$ on $\Omega \cap \{\epsilon/2 \leq x \leq \epsilon\}$, we have

$$v = 0 \quad \text{on} \quad \Omega_\epsilon \cap \{x = \epsilon\}.$$

Therefore, by Maximum principle we finally get $v > 0$ near the sonic arc as desired.

$\sharp$

Proposition 4.9 (estimation from above)**.** *Almost we can get $\psi_x > 0$ in Ω_ϵ*

Remark 4.10. *Precisely to claim this Proposition is $\psi_x > -\delta x$ where $\delta > 0$ can be arbitrary small if we keep close to sonic arc (depending on δ).*

Proof for Proposition 4.9 Here I directly explain ideas of the proof given by Chen-Feldman in [10]. They obtain this Proposition in two steps:

Step 1. $\psi_\eta \leq 0$ in the whole domain Ω.

For this, we can differentiate equation with respect to η, and differentiating boundary conditions on shock and wedge in the tangential direction and using equation, we derive homogeneous elliptic equation, and homogeneous oblique boundary conditions on shock and wedge for $w = \psi_\eta$. Also we know that $w < 0$ on symmetry line by the boundary condition $\psi_\eta = -v_2$ on symmetry line (this is main point, we propagate the negativity of ψ_η from the symmetry line), and $w = 0$ on sonic arc since from Proposition 4.6, $D\psi = 0$ on sonic arc. Then we get $\psi_\eta \leq 0$ by maximum principle.

Step 2. $\psi_x \geq -\frac{4}{3(\gamma+1)}x$ in Ω_ϵ.

For this, we first express ψ_x through ψ_y and ψ_η, and show that the estimate $|\psi_y| \leq$

$Cx^{3/4}$ in Proposition 4.6, combined with negativity of ψ_η allows to get

$$\psi_x = -\frac{1}{\sin\theta}\psi_\eta + \frac{\cot\theta}{r}\psi_y \geq \frac{\cot\theta}{r}\psi_y \geq -C|\psi_y|, \tag{4-24}$$

where the polar angle θ satisfies $\sin\theta = \frac{\eta}{\sqrt{\xi^2+\eta^2}} > 0$ and thus we can get the estimate from below. $\sharp$

Chapter 5 Higher regularity near sonic arc away from shock

In this chapter, we will also use comparison principle to get the higher regularity $C^{2,\alpha}$ near sonic arc except P_0, which in fact holds only for potential flow and fails for general Euler flow. For achieving this result, we need a quadratic lower bound for ψ (see Section 5.1). Then careful compare the boundary conditions in order to use Maximum principle to get a supersolution (see Section 5.2).

The following standard comparison principle for the operator L follows from[24].

Lemma 5.1 (comparison principle). *Let $\Omega \subset R^2$ be an open bounded set. Let $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that the operator L is elliptic in Ω with respect to either u or v. Let $Lu \leq Lv$ in Ω and $u \geq v$ on $\partial\Omega$. Then $u \geq v$ in Ω.*

Go back to equation (4-5)

$$(2x - (\gamma + 1)\psi_x)\psi_{xx} + \frac{1}{c_2}\psi_{yy} - \psi_x = 0,$$

noting it is elliptic with respect to ψ in $\{x > 0\}$ if $\psi_x < \frac{2x}{\gamma+1}$. In this section, we consider the solution satisfying

$$-Mx \leq \psi_x \leq \frac{2-\beta}{\gamma+1}x \quad \text{in} \quad \{x > 0\} \tag{5-1}$$

for some constant $M \geq 0$ and $\beta \in (0, 1)$, where the lower and upper bound come from section 3.3 and 3.4 respectively. Then the equation (4-5) is uniformly elliptic in every sundomain of $\{x > \delta\}$ with $\delta > 0$. The same is true for the full equation (4-4).

5.1 Quadratic lower bound of ψ

We have note the equation (4-5) is uniformly elliptic and thus $C^{2,\alpha}$ for all $\alpha \in (0, 1)$ with respect to ψ inside $x > \delta$ with $\delta > 0$. Thus, the idea here is to construct a positive subsolution of (4-5), which turns out to provide a lower bound of ψ.

Here we will just consider on ordinary differential equation level in the direction x. In fact this is a good approximation away from P_1 but both variables x, y will play a role near P_1. We keep only main term in the equation (comparing with (36)):

$$N[\psi] = [2x - (\gamma + 1)\psi_x]\psi_{xx} - \psi_x = 0 \quad \text{in} \quad \{x \in (0, \epsilon)\} \tag{5-2}$$

$$\psi(0) = 0, \psi(\epsilon) = \lambda > 0$$

and our ϵ need to be small in order to control the extra terms of the equation.

Proposition 5.2. *Solution ψ in equation (5-2) satisfies*

$$\psi \geq \frac{\mu}{2(\gamma + 1)}x^2, \quad \textit{for some small } \mu > 0. \tag{5-3}$$

Proof for Proposition 5.2. Set $u = \frac{1}{2(\gamma+1)}x^2$. We calculate $u_x = \frac{\mu}{\gamma+1}x, u_{xx} = \frac{\mu}{\gamma+1}$ and thus

$$N(u) = \frac{\mu}{\gamma + 1}(1 - \mu)x > 0$$

Also function N is elliptic for u on $(0, \epsilon)$ since $2x - (\gamma + 1)u_x = x(2 - \mu) > 0$. One can note that on the boundaries: $\psi(0) = 0 = u(0)$ and $\psi(\epsilon) = \lambda > u(\epsilon)$ if μ is small. Thus by Lemma 1, $\psi \geq u = \frac{\mu}{2(\gamma+1)}x^2$ on $(0, \epsilon)$. $\sharp$

5.2 $C^{2,\alpha}$ estimate of ψ

If ψ satisfies (4-5), boundary conditions and (5-1), then it is expected to be very close to $\frac{x^2}{2(\gamma+1)}$. Note $\frac{x^2}{2(\gamma+1)}$ in fact is a solution to (4-5). More precisely, we now to show

Theorem 5.3. $\psi - v = O(x^{2+\alpha})(\overline{\Omega})$, *where* $v = \frac{1}{2(\gamma+1)}x^2$.

Proof for Theorem 5.3. In order to obtain this result, we study the function

$$W = v - \psi.$$

Calculate $W_x = \frac{x}{\gamma+1} - \psi_x$, $W_{xx} = \frac{1}{\gamma+1} - \psi_{xx}$ and again by (5-2), we have

$$N^*[W] := [x + (\gamma+1)W_x]W_{xx} - 2W_x = 0 \quad \text{in} \quad \{0 < x < \epsilon_1\}. \tag{5-4}$$

Moreover, since $\psi_x \le \frac{2-\beta}{\gamma+1}x$ with $\psi(0) = 0$, we do calculus and get $\psi \le \frac{2-\beta}{2(\gamma+1)}x^2$. Combine with inequality (5-3), we get an estimate for W:

$$\frac{\beta - 1}{2(\gamma+1)} \le W(x) \le \frac{1-\mu}{2(\gamma+1)}x^2, \tag{5-5}$$

where $0 < \beta < 1$.

Then, let $\widehat{W} = \frac{\tau}{\gamma+1}x^{2+\alpha}$, where $\alpha \in (0,1), \tau > 0$ and we will show $\widehat{W}$ is a super-solution for N^*. First, plug $\widehat{W}$ into $N^*[\widehat{W}]$ to get

$$N^*[\widehat{W}] = \frac{2+\alpha}{\gamma+1}\tau x^{1+\alpha}(\alpha - 1 + (1+\alpha)(2+\alpha)\tau x^\alpha)$$

on $(0, \epsilon_1)$, here the coefficient $\frac{2+\alpha}{1+\gamma}\tau > 0$ by definition. Moreover,

1) the equation

$$N^*[\widehat{W}] \le 0$$

if $(1+\alpha)(2+\alpha)\tau\epsilon_1^\alpha\alpha \le 1 - \alpha$. So we only need to choose $\tau = \frac{1-\alpha}{(1+\alpha)(2+\alpha)}\epsilon_1^{-\alpha}$.

On boundaries we also need to have

2) $\widehat{W}(\epsilon_1) \ge W(\epsilon_1)$, that is

$$W(\epsilon_1) \le \frac{(1-\alpha)\epsilon_1^2}{(1+\alpha)(2+\alpha)(\gamma+1)}.$$

Note we have a upper bound for W indicated in (5-5), we choose α to satisfy

$$\frac{(1-\alpha)}{(1+\alpha)(2+\alpha)} = \frac{1-\mu}{2},$$

with small $\mu > 0$ got from above. Then calculate $\alpha = \frac{3\mu - 5 + \sqrt{\mu^2 - 22\mu + 25}}{2(1-\mu)} \in (0,1)$ as desired.

At last, we also need N^* to be elliptic for $\widehat{W} \in (0, \epsilon_1)$

3) in fact the second-order coefficient $x + (\gamma+1)\widehat{W}_x \geq x > 0$. Thus the ellipticity follows obviously.

Therefore $\widehat{W}$ is a supersolution for N^* and by Lemma 5.1, we would get $W \leq \widehat{W}$ on $(0, \epsilon_1)$. Similarly in order to show $W \geq -\widehat{W}$ on $(0, \epsilon_1)$, we also follow the steps above with τ, α determined by β instead of μ according to the lower bound indicated in (5-5).

Therefore

$$\left|\psi - \frac{1}{2(\gamma+1)}x^2\right| \leq |\widehat{W}| \leq Cx^{2+\alpha},$$

which is exactly our desired result. $\sharp$

Chapter 6 Conclusion and open problems

6.1 Conclusion

In this thesis, we mainly do two works. The first is to systematically and rigorously formulate the shock reflection phenomena by a free boundary model. Especially, we extend our talk to three details: (1) explain the ideas behind different pattern-transition criteria; (2) discuss the similarity and difference between potential flow and Euler flow for estimation near sonic arc; (3) talk about the idea of introducing ellipticity cutoff with different choice of constant.

Second and also the primary work we do in the Chapter 4 and Chapter 5 is to give simple and direct proofs for classical estimation of $\psi = \varphi - \varphi_2$ by developed Maximum principle adapted to free boundary problems. Also we use the comparison principle and scaling techniques for elliptic equations to reprove $C^{1,1}$-regularity up to sonic arc and $C^{2,\alpha}$-regularity up to the sonic arc away from shock point P_1. And we point out the optimal regularity near sonic arc is particularly holding for the potential flow.

6.2 Open problems

As we mentioned in Chapter 1, there is still a long journey to fully understand the shock reflection problem. For example,

(1) Uniqueness of regular reflection solution. So far, some Mathematician believe this result depends on geometric properties of the shock and convincingly the convexity would be sufficient.

(2) Prove all these results for Euler system. Difficulty here involves the vorticity blowup near stagnation points, which is noticed by D. Serre for isentropic Euler system[25].

(3) Prove all there results for Mach reflection.

Furthermore, a good solution to the shock reflection problem not only provides us the understanding for shock reflection-diffraction phenomena and behavior of entropy solutions to multidimensional hyperbolic systems of conservation laws, but also provides us important new ideas and techniques for overcoming the core challenges in multidimensional problems in conservation laws and other branches in nonlinear partial differential equations. Moreover, the shock reflection problem also serves as a great test problem used in examining the approaches developed in nonlinear partial differential equations and numerical examples.

I would like to continue my study and research in these aspects after my graduation.

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